Static Stellar Structure

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Most of the Life of A Star is Spent in Equilibrium

- Evolutionary Changes are generally slow and can usually be handled in a quasistationary manner
- We generally assume:
  - Hydrostatic Equilibrium
  - Thermodynamic Equilibrium
- The Equation of Hydrodynamic Equilibrium

$$\rho \frac{d^2 r}{dt^2} = -\frac{Gm(r)\rho}{r^2} - \frac{\partial P}{\partial r}$$

Limits on Hydrostatic Equilibrium

- If the system is not "Moving" - accelerating in reality - then $d^2r/dt^2 = 0$ and then one recovers the equation of hydrostatic equilibrium:

$$-\frac{Gm(r)\rho}{r^2} = \frac{\partial P}{\partial r} = \frac{dP}{dr}$$

- If $\partial P/\partial r \sim 0$ then which is just the freefall condition for which the time scale is $t_{ff} \simeq (GM/R^3)^{-1/2}$

$$\frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2}$$

Dominant Pressure Gradient

- When the pressure gradient dP/dr dominates one gets $(r/t)^2 \sim P/\rho$
  - This implies that the fluid elements must move at the local sonic velocity: $c_s = \partial P/\partial \rho$.
- When hydrostatic equilibrium applies
  - $V \ll c_s$
  - $t_e \gg t_{ff}$ where t_e is the evolutionary time scale

Hydrostatic Equilibrium

- Consider a spherical star
  - Shell of radius r, thickness dr and density ρ(r)
- Gravitional Force: ↓ $(Gm(r)/r^2)\ 4\pi r^2\rho(r)dr$
- Pressure Force: ↑ $4r^2dP$ where dP is the pressure difference across dr
- Equate the two: $4\pi r^2 dP = (Gm(r)/r^2)\ 4\pi r^2\rho(r)dr$
- $r^2 dP = Gm(r)\ \rho(r)dr$
- $dP/dr = -\rho(r)(Gm(r)/r^2)$
- The - sign takes care of the fact that the pressure decreases outward.

Mass Continuity

- m(r) = mass within a shell = $m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'$
- This is a first order differential equation which needs boundary conditions
  - We choose P_c = the central pressure.
- Let us derive another form of the hydrostatic equation using the mass continuity equation.
  - Express the mass continuity equation as a differential: $dm/dr = 4\pi r^2 \rho(r)$.
  - Now divide the hydrostatic equation by the mass-continuity equation to get: $dP/dm = Gm/4\pi r^4(m)$

The Hydrostatic Equation in Mass Coordinates

- $dP/dm = Gm/4\pi r^4(m)$
  - The independent variable is m
  - r is treated as a function of m
  - The limits on m are:
  - 0 at r = 0
  - M at r = R (this is the boundary condition on the mass equation itself).
- Why?
  - Radius can be difficult to define
  - Mass is fixed.

The Central Pressure

- Consider the quantity: $P + Gm(r)^2/8\pi r^4$
- Take the derivative with respect to r:

$$\frac{d}{dr}\left[P + \frac{Gm(r)^2}{8\pi r^4}\right] = \frac{dP}{dr} + \frac{Gm(r)}{4\pi r^4}\frac{dm}{dr} - \frac{Gm(r)^2}{2\pi r^5}$$

- But the first two terms are equal and opposite so the derivative is $-Gm^2/2r^5$.
- Since the derivative is negative *it must decrease outwards*.
- At the center $m^2/r^4 \rightarrow 0$ and $P = P_c$. At $r = R$ $P = 0$ therefore $P_c > GM^2/8\pi R^4$

The Virial Theorem

$$\frac{dP}{dm} = -\frac{mG}{4\pi r(m)^4}$$

$$4\pi r^3 \frac{dP}{dm} = -\frac{mG}{r}$$

$$\frac{d}{dm}(4\pi r^3 P) - 4\pi r^2 3P \frac{dr}{dm} = -\frac{mG}{r}$$

$$(4\pi r^3 P)\,\Big|_0^M - \int_0^M \frac{3P}{\rho} dm = -\int_0^M \frac{mG}{r} dm$$

Remember: $4\pi r^2 r dr = dm$

The Virial Theorem

- The term $(4\pi r^3 P)\big|_0^M$ is 0: r(0) = 0 and P(M) = 0
- Remember that we are considering P, ρ, and r as variables of m
- For a non-relativistic gas: 3P/ = 2 * Thermal energy per unit mass.

$$\therefore \quad \int_0^M \frac{3P}{\rho} dm = 2U \quad \textit{for the entire star}$$

$$\therefore \quad \int \frac{GM}{r} dm = \Omega \quad \textit{the gravitional binding energy}$$

The Virial Theorem

- $-2U = \Omega$
- $2U + \Omega = 0$ Virial Theorem
- Note that $E = U + \Omega$ or that $E+U = 0$
- This is only true if "quasistatic." If hydrodynamic then there is a modification of the Virial Theorem that will work.

The Importance of the Virial Theorem

- Let us collapse a star due to pressure imbalance:
  - This will release $-\Delta\Omega$
- If hydrostatic equilibrium is to be maintained the thermal energy must change by:
  - $\Delta U = -1/2\ \Delta\Omega$
- This leaves $1/2\ \Delta\Omega$ to be “lost” from star
  - Normally it is radiated

What Happens?

- Star gets hotter
- Energy is radiated into space
- System becomes more tightly bound: E decreases
- Note that the contraction leads to H burning (as long as the mass is greater than the critical mass).

An Atmospheric Use of Pressure

- We use a different form of the equation of hydrostatic equilibrium in an atmosphere.
- The atmosphere's thickness is small compared to the radius of the star (or the mass of the atmosphere is small compared to the mass of the star)
  - For the Sun the photosphere depth is measured in the 100's of km whereas the solar radius is 700,00 km. The photosphere mass is about 1% of the Sun.

Atmospheric Pressure

- Geometry: Plane Parallel
- $dP/dr = -Gm(r)\rho/r^2$ (Hydrostatic Equation)
  - $R \approx r$ and $m(R) = M$. We use h measured with respect to some arbitrary 0 level.
- $dP/dh = -g\rho$ where g = acceleration of gravity. For the Sun log(g) = 4.4 (units are cgs)
- Assume a constant T in the atmosphere. $P = nkT$ and we use $n = \rho/\mu m_H$ (μ is the mean molecular weight) so
- $$P = \rho kT/\mu m_H$$

Atmospheric Pressure Continued

- $P = \rho kT/\mu m_H$
- $dP = -g\ \rho\ dh$
- $dP = -g\ P(\mu m_H/kT)\ dh$
- Or $dP/P = -g(\mu m_H/kT)\ dh$

$$Integrate: \quad P = P_0 e^{-\frac{g m_H \mu h}{kT}}$$

$$\rho = \rho_0 e^{-\frac{g m_H \mu h}{kT}}$$

- Where: $P_0 = P$ and $\rho_0 = \rho$ at $h = 0$.

Scale Heights

- H (the scale height) is defined as $kT / \mu m_H g$
  - It defines the scale length by which P decreases by a factor of e.
- In non-isothermal atmospheres scale heights are still of importance:
  - $H \equiv - (dP/dh)^{-1} P = -(dh/d(\ln P))$

Simple Models

To do this right we need data on energy generation and energy transfer but:

- The linear model: $\rho = \rho_c(1\text{-}r/R)$
- Polytropic Model: $P = K\rho^\gamma$
  - K and γ independent of r
  - γ not necessarily c_p/c_v

The Linear Model

$$\rho = \rho_c(1 - \frac{r}{R})$$

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho_c(1 - \frac{r}{R})$$

- Defining Equation
- Put in the Hydrostatic Equation
- Now we need to deal with m(r)

An Expression for m(r)

$$\frac{dm}{dr} = 4\pi r^2 \rho_c (1 - \frac{r}{R})$$

Equation of Continuity

$$= 4\pi r^2 \rho_c - \frac{4\pi r^3 \rho_c}{R}$$

$$m(r) = \frac{4\pi r^3 \rho_c}{3} - \frac{\pi r^4 \rho_c}{R}$$

Integrate from 0 to r

$$Total\ Mass\ M = m(R) = \frac{\pi R^3 \rho_c}{3}$$

$$So \quad m(r) = M\left(\frac{4r^3}{R^3} - \frac{3r^4}{R^4}\right)$$

Linear Model

- We want a particular (M,R)
- $\rho_c = 3M/\pi R^3$ and take $P_c = P(\rho_c)$
- Substitute m(r) into the hydrostatic equation:

$$\frac{dP}{dr} = \frac{\pi G}{r^2}\rho_c{}^2\left(\frac{4r^3}{3} - \frac{r^4}{R}\right)\left(1 - \frac{r}{R}\right)$$

$$= -\pi G\rho_c{}^2\left(\frac{4r}{3} - \frac{7r^2}{3R} + \frac{r^3}{R^2}\right)$$

$$So \quad P = P_c - \pi G\rho_c{}^2\left(\frac{2r^2}{3} - \frac{7r^3}{9R} + \frac{r^4}{4R^2}\right)$$

where P_c is the central pressure.

Central Pressure

$$At\ the\ surface\ r = R\ and\ \ P = 0$$

$$P(R) = P_c - \frac{5\pi G\rho_c^2 R^2}{36} = 0$$

$$\therefore P_c = \frac{5\pi G\rho_c^2 R^2}{36} = \frac{5\pi G R^2}{36}\frac{9M^2}{\pi^2 R^6}$$

$$P_c = \frac{5GM^2}{4\pi R^4}$$

The Pressure and Temperature Structure

- The pressure at any radius is then:

$$P(r) = \frac{5\pi}{36} G \rho_c^2 R^2 \left(1 - \frac{24r^2}{5R^2} + \frac{28r^3}{5R^3} - \frac{9r^4}{5R^4} \right)$$

- Ignoring Radiation Pressure:

$$T = \frac{\mu m_H P}{k\rho}$$

$$= \frac{5\pi}{36} \frac{G \mu m_H}{k} \rho_c R^2 \left(1 + \frac{r}{R} - \frac{19r^2}{5R^2} + \frac{9r^3}{5R^3} \right)$$

Polytropes

- $P = K\rho^{\gamma}$ which can also be written as
- $P = K\rho^{(n+1)/n}$
  - N ≡ Polytropic Index
- Consider Adiabatic Convective Equilibrium
  - Completely convective
  - Comes to equilibrium
  - No radiation pressure
  - Then $P = K\rho^{\gamma}$ where $\gamma = 5/3$ (for an ideal monoatomic gas) ==> $n = 1.5$

Gas and Radiation Pressure

- $P_g = (N_0 k/\mu)\rho T = \beta P$
- $P_r = 1/3\ a\ T^4 = (1-\beta)P$
- $P = P$
- $P_g/\beta = P_r/(1-\beta)$
- $1/\beta(N_0 k/\mu)\rho T = 1/(1-\beta)\ 1/3\ a\ T^4$
- Solve for T: $T^3 = (3(1-\beta)/a\beta)\ (N_0 k/\mu)\rho$
- $T = ((3(1-\beta)/a\beta)\ (N_0 k/\mu))^{1/3}\ \rho^{1/3}$

The Pressure Equation

$$P = \frac{N_0 k}{\mu} \frac{\rho T}{\beta}$$

$$= \left[\left(\frac{N_0 k}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta} \right]^{\frac{1}{3}} \rho^{\frac{4}{3}}$$

- True for each point in the star
- If β ≠ f(R), i.e. is a constant then
- $P = K\rho^{4/3}$
- n = 3 polytrope or γ = 4/3

The Eddington Standard Model n = 3

- For n = 3 $\rho \propto T^3$
- The general result is: $\rho \propto T^n$
- Let us proceed to the Lane-Emden Equation
- $\rho \equiv \lambda\varphi^n$
  - λ is a scaling parameter identified with ρ_c - the central density
  - φ^n is normalized to 1 at the center.
  - Eqn 0: $P = K\rho^{(n+1)/n} = K\lambda^{(n+1)/n}\varphi^{n+1}$
  - Eqn 1: Hydrostatic Eqn: $dP/dr = -G\rho m(r)/r^2$
  - Eqn 2: Mass Continuity: $dm/dr = 4\pi r^2\rho$

Working Onward

$$Solve\,1\ \ for\ m(r): \quad m(r) = -\frac{r^2}{\rho G}\frac{dP}{dr}$$

$$Put\ m(r)\ in\ 2: \quad \frac{1}{r^2}\left[\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)\right] = -4\pi G\rho$$

$$\frac{dP}{dr} = K\lambda^{\frac{n+1}{n}}(n+1)\phi^n\frac{d\phi}{dr}$$

$$\frac{1}{r^2}\left[\frac{d}{dr}\left(\frac{r^2}{\lambda\phi^n}K\lambda^{\frac{n+1}{n}}(n+1)\phi^n\frac{d\phi}{dr}\right)\right] = -4\pi G\lambda\phi^n$$

Simplify

$$\frac{1}{r^2}\left[\frac{d}{dr}\left(\frac{r^2}{\lambda\phi^n}K\lambda^{\frac{n+1}{n}}(n+1)\phi^n\frac{d\phi}{dr}\right)\right]=-4\pi G\lambda\phi^n$$

$$K\lambda^{\frac{1}{n}}(n+1)\frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)\right]=-4\pi G\lambda\phi^n$$

$$K\lambda^{\frac{1-n}{n}}(n+1)\frac{1}{4\pi G}\frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)\right]=-\phi^n$$

$$Define\ a\equiv\left[\frac{K\lambda^{\frac{1-n}{n}}(n+1)}{4\pi G}\right]^{\frac{1}{2}}$$

$$\xi\equiv\frac{r}{a}\quad so\quad a\,d\xi=dr$$

$$So:\quad \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right)=-\phi^n$$

The Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n$$

- Boundary Conditions:
  - Center ξξ = 0; φ = 1 (the function); dφ/d ξ = 0
- Solutions for n = 0, 1, 5 exist and are of the form:
- $\varphi(\xi) = C_0 + C_2\,\xi^2 + C^4\,\xi^4 + \ldots$
- $= 1 - (1/6)\,\xi^2 + (n/120)\,\xi^4 - \ldots$ for $\xi = 1$ $(n>0)$
- For n < 5 the solution decreases monotonically and φ → 0 at some value ξ_1 which represents the value of the boundary level.

General Properties

- For each n with a specified K there exists a family of solutions which is specified by the central density.
  - For the standard model:

$$K = \left[\left(\frac{N_0 k}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{\frac{1}{3}}$$

- We want Radius, m(r), or m(ξ)
- Central Pressure, Density
- Central Temperature

Radius and Mass

$$R = a\xi_1$$

$$= \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \lambda^{\frac{1-n}{2n}} \xi_1$$

$$M(\xi) = \int_0^{a\xi} 4\pi r^2 \rho dr$$

$$= 4\pi a^3 \int_0^{\xi} \lambda \phi^n \xi^2 d\xi$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) = -\phi^n$$

$$M(\xi) = -4\pi a^3 \lambda \int_0^{\xi} \frac{d}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) d\xi$$

$$M(\xi) = -4\pi a^3 \lambda \int_0^{\xi} d\left(\xi^2 \frac{d\phi}{d\xi}\right)$$

$$= -4\pi a^3 \lambda \xi^2 \frac{d\phi}{d\xi}$$

Now substitute for a and evaluate at $\xi = \xi_1$ *(boundary)*

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \lambda^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$$

The Mean to Central Density

$$\rho_c = \lambda$$

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{-4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \lambda^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi=\xi_1}}{\frac{4}{3}\pi\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \lambda^{\frac{3(1-n)}{2n}} \xi_1^3}$$

$$\bar{\rho} = -\frac{3}{\xi_1}\lambda\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$$

$$\frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1}\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$$

The Central Pressure

- At the center $\xi = 0$ and $\varphi = 1$ so $P_c = K\lambda^{n+1/n}$
- This is because $P = K\rho^{n+1/n} = K\lambda^{(n+1)/n}\varphi^{n+1}$
- Now take the radius equation:

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}} \lambda^{\frac{1-n}{2n}} \xi_1$$

$$= \left[\frac{(n+1)\xi_1^{\,2}}{4\pi G}\right]^{\frac{1}{2}} \left[K\lambda^{\frac{1-n}{n}}\right]^{\frac{1}{2}}$$

$$So \quad K\lambda^{\frac{1-n}{n}} = \frac{4\pi G R^2}{n+1)\xi_1^{\,2}}$$

Central Pressure

$$P_c = K\lambda^{\frac{1-n}{n}}\lambda^2 = K\lambda^{\frac{1-n}{n}}\rho_c^2$$

$$= \frac{4\pi G R^2}{(n+1)\xi^2}\left[\frac{\xi_1}{3}\frac{1}{\left(d\phi/d\xi\right)_{\xi_1}}\right]^2 \bar{\rho}^2$$

$$But \quad M = \frac{4}{3}\pi R^3 \bar{\rho} \quad so \quad M^2 = \frac{16}{9}\pi^2 R^6 \bar{\rho}^2$$

$$P_c = \frac{4\pi G R^2}{(n+1)\xi^2}\left[\frac{\xi_1}{3}\frac{1}{\left(d\phi/d\xi\right)_{\xi_1}}\right]^2 \left[\frac{9M^2}{16\pi^2 R^6}\right]$$

$$= \frac{GM^2}{4\pi R^4 (n+1)}\left[\frac{1}{\left(d\phi/d\xi\right)_{\xi_1}}\right]^2$$

Central Temperature

$$P_g = \frac{N_0 k}{\mu} \rho_c T_c = \beta_c P_c$$

which means

$$T_c = \frac{\beta_c P_c \mu}{N_0 k \rho_c}$$