Line Broadening and Opacity

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Total Absorption Coefficient $=\kappa_c + \kappa_l$

- Absorption Processes: Simplest Model
  - Photon absorbed from forward beam and reemitted in arbitrary direction
    - BUT: this could be scattering!
- Absorption Processes: Better Model
  - The reemitted photon is part of an energy distribution characteristic of the local temperature
    - $B\nu_\nu(T_c) > B_\nu(T_l)$
- “Some” of $B_\nu(T_c)$ has been removed and $B_\nu(T_l)$ is less than $B_\nu(T_c)$ and has arbitrary direction.

Doppler Broadening

Non Relativistic: ❍ν/ν_0 = v/c

- The emitted frequency of an atom moving at v will be ν′: $\nu' = \nu_0 +$ ❍ν $= \nu_0 + (v/c)\nu_0$
- The absorption coefficient will be:

$$a_\nu = \frac{\pi e^2}{m_e c} f \frac{\Gamma}{4\pi^2} \frac{1}{(\nu' - \nu)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

- Where ν = Frequency of Interest
- ν′ = Emitted Frequency
- ν_0 = Rest Frequency

Total Absorption

Per atom in the unit frequency interval at ν

- Multiply by the fraction of atoms with velocity v to v + dv: The Maxwell Boltzman distribution gives this:

$$\frac{dN}{N} = \sqrt{\frac{M}{2\pi kT}}e^{\frac{-M\mathbf{v}^2}{2kT}}d\mathbf{v}$$

  – Where $M = AM_0$ = mass of the atom (A = atomic weight and M_0 = 1 AMU)

- Now integrate over velocity

$$a_\nu = \frac{\pi e^2}{m_e c} f \frac{\Gamma}{4\pi^2} \int_{-\infty}^{\infty} \frac{\sqrt{\frac{M}{2\pi kT}}e^{\frac{-M\mathbf{v}^2}{2kT}}}{(\nu_0 + \frac{\mathbf{v}}{c}\nu_0 - \nu)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} d\mathbf{v}$$

This is a Rather Messy Integral

$$\Delta\nu_0 = \frac{\nu_0}{c}\sqrt{\frac{2kT}{M}}$$

$$\Delta\nu = \frac{\nu_0}{c}\mathbf{v}$$

Doppler Equation

$$d\mathbf{v} = \frac{c}{\nu_0}d(\Delta\nu)$$

$$y = \frac{\Delta\nu}{\Delta\nu_0}$$

$$u = \frac{\nu - \nu_0}{\Delta\nu_0}$$

$$\delta' = \frac{\Gamma}{4\pi}$$

Gamma is the effective

$$a = \frac{\delta'}{\Delta\nu_0}$$

Simplify!

$$a_\nu = \frac{\pi e^2}{m_e c} f \frac{\Gamma}{4\pi^2} \int_{-\infty}^{\infty} \frac{\sqrt{\frac{M}{2\pi kT}} e^{\frac{-M\mathbf{v}^2}{2kT}}}{(\nu_0 + \frac{\mathbf{v}}{c}\nu_0 - \nu)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} d\mathbf{v}$$

$$\frac{M\mathbf{v}^2}{2kT} \textbf{ and } \Delta\nu_0 = \frac{\nu_0}{c}\sqrt{\frac{2kT}{M}}$$

$$\frac{\mathbf{c}^2 \Delta\nu_0{}^2}{\nu_0{}^2} = \frac{2kT}{M}$$

$$\Delta\nu^2 = \left(\frac{\nu_0}{c}\right)^2 \mathbf{v}^2$$

$$-\frac{M\mathbf{v}^2}{2kT} = \frac{-\nu_0{}^2 \mathbf{v}^2}{\mathbf{c}^2 \Delta\nu_0{}^2} = -\left(\frac{\Delta\nu^2}{\Delta\nu_0{}^2}\right) = -y^2$$

$$\left[\frac{\Gamma}{4\pi}\right]^2 = \delta'^2 = a^2 \Delta\nu_0^2$$

$$(\nu_0 + \frac{\mathbf{v}}{c}\nu_0 - \nu)^2 = \Delta\nu_0^2 (u - y)^2$$

Our Equation is Now

$$\int \frac{e^{-y^2}}{\Delta \nu_0^2 a^2 + \Delta \nu_0^2 (u-y)^2} d\mathbf{v}$$

- Note that $\Delta\nu_0$ is a constant = $(\nu_0/c)\,(\sqrt{2kT/M})$ so pull it out
- What about dv: $dv = (c/\nu_0)d(\Delta\nu)$
- $y = \Delta\nu/\Delta\nu_0$
- $dy = (1/\Delta\nu_0)\, d(\Delta\nu)$
- $dv = (c\Delta\nu_0/\nu_0)dy$
- So put that in

Getting Closer

$$\frac{c}{\nu_0 \Delta\nu_0} \int \frac{e^{-y^2}}{a^2 + (u-y)^2} dy$$

Constants: First the constant in the integral

$$\left(\frac{M}{2\pi kT}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \left(\frac{M}{2kT}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \frac{\nu_0}{c\Delta\nu_0}$$

So then: $$\frac{\pi e^2}{m_e c} f \frac{\Gamma}{4\pi^2} \frac{1}{\sqrt{\pi}} \frac{\nu_0}{c\Delta\nu_0} \frac{c}{\nu_0 \Delta\nu_0} = \frac{\pi e^2}{m_e c} f \frac{\Gamma}{4\pi^2} \frac{1}{\sqrt{\pi}} \frac{1}{\Delta\nu_0{}^2}$$

$$\delta' \equiv \frac{\Gamma}{4\pi} \quad and \quad a \equiv \frac{\delta'}{\Delta\nu_0} \quad so$$

$$\frac{\pi e^2}{m_e c} f \frac{1}{\pi\sqrt{\pi}} \frac{1}{\Delta\nu_0} \frac{\delta'}{\Delta\nu_0} = \frac{\pi e^2}{m_e c} f \frac{1}{\sqrt{\pi}} \frac{1}{\Delta\nu_0} \frac{a}{\pi} = \alpha_0 \frac{a}{\pi}$$

Continuing On

- Note the α_0 does not depend on frequency (except for ν_0 = constant for any line; Oν_0 = $(\nu_0/c)\sqrt{(2kT/M)}$). It is called the absorption coefficient at line center.
- The normalized integral H(a,u) is called the Voigt function. It carries the frequency dependence of the line.

$$H(a,u) = \frac{a}{\pi}\int \frac{e^{-y^2}}{a^2 + (u-y)^2}dy$$

Pulling It Together

The Line Opacity α_l

$$\kappa_l = N\frac{\pi e^2}{m_e c} f \frac{1}{\sqrt{\pi}} \frac{1}{\Delta\nu_0} H(a,u)\left[1 - e^{\frac{-h\nu}{kT}}\right]$$

Be Careful about the $1/\sqrt{\pi}$ as it can be taken up in the normalization of H(a,u).

The Terms

- a is the broadening term (natural, etc)
  - $a = \delta'/O\nu_0 = (\Gamma/4\pi)/((\nu_0/c)\sqrt{(2kT/M)})$
- u is the Doppler term
  - $u = \nu - \nu_0/((\nu_0/c)\sqrt{(2kT/M)})$
- At line center u = 0 and α_0 is the absorption coefficient at line center so H(a,0) = 1 in this treatment. Note that different treatments give different normalizations.

Wavelength Forms

$$u = \frac{\Delta\lambda}{\left(\frac{\lambda}{c}\left[\frac{2kT}{M} + \xi^2\right]^{\frac{1}{2}}\right)}$$

$$a = \frac{\Gamma_n + \Gamma_S + \Gamma_W}{2\pi\lambda^{-1}\left[\frac{2kT}{M} + \xi^2\right]^{\frac{1}{2}}}$$

- Γ′s are broadenings due to various mechanisms
- ε is any additional microscopic motion which may be needed.

Simple Line Profiles

- First the emergent continuum flux is:

$$F_\lambda(0) = 2\int_0^\infty S_\lambda(\tau_\lambda) E_2(\tau_\lambda) d\tau_\lambda$$

- The source function $S_\lambda(\tau_\lambda) = B_\lambda(\tau_\lambda)$.
- $E_2(\tau_\lambda)$ = the second exponential integral.
- The total emergent continuum flux is:

$$F'_\lambda(0) = 2\int_0^\infty S'_\lambda(\tau_\lambda + \tau_l) E_2(\tau_\lambda + \tau_l) d(\tau_\lambda + \tau_l)$$

- τ_λ refers to the continuous opacities and τ_l to the line opacity

The Residual Flux R(❍λ)

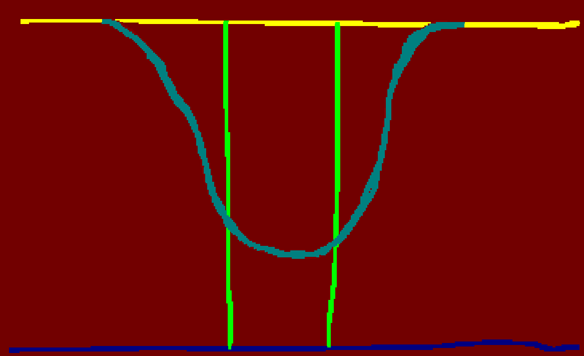
$$R(\Delta\lambda) = \frac{2}{F_\lambda(0)} \int_0^\infty S'_\lambda(\tau_\lambda + \tau_l) E_2(\tau_\lambda + \tau_l) d(\tau_\lambda + \tau_l)$$

- This is the ratioed output of the star
  - 0 = No Light
  - 1 = Continuum
- We often prefer to use depths: D(❍λ) = 1 - R(❍λ)

The Equivalent Width

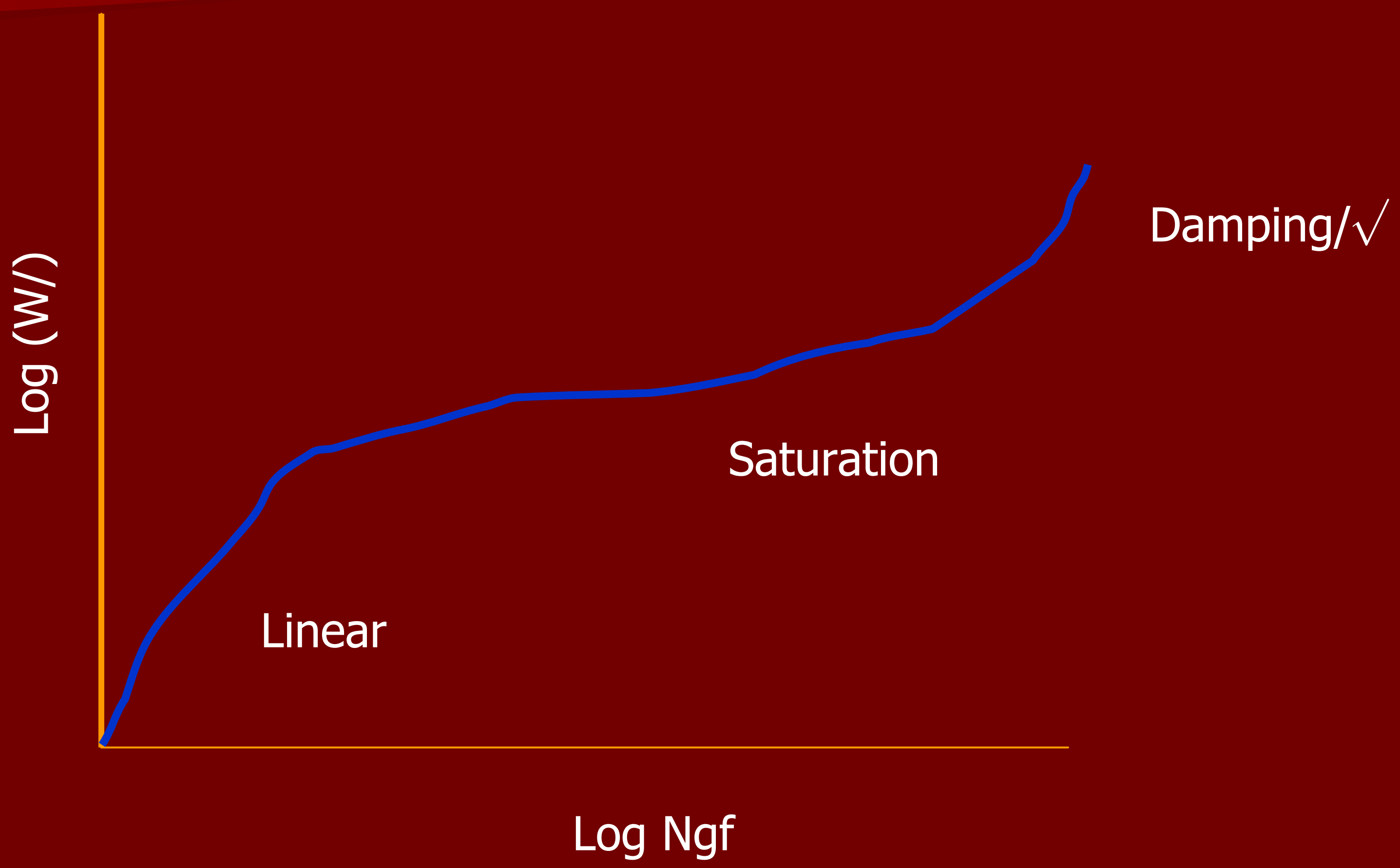
- Often if the lines are not closely spaced one works with the equivalent width:

$$W_\lambda = \int_{-\infty}^{\infty} D(\Delta\lambda)\,d\Delta\lambda$$



- W_λ is usually expressed in mÅ
- W_λ = 1.06(❍λ)D(❍λ) for a gaussian line. ❍λ is the FWHM.

Curve of Growth



The Cookbook

To Compute a Line You Need

- Model Atmosphere: (τ_R, T, P_{gas}, N_e, K_R)
- Atomic Data
  - Wavelength/Frequency of Line
  - Excitation Potential
  - gf
  - Species (this specifies U(T) and ionization potential)
- Abundance of Element (Initial)

What Do You Do Now

- τ_R,K_R: Optical depth and opacity are specified at some reference wavelength or may be Rosseland values.
- The wavelength of interest is somewhere else: λ
- So you need α_λ : You need T, N_e, and P_g and how to calculate f-f, b-f, and b-b but in computing lines one does not add b-b to the continuous opacity.

$$\frac{\tau_\lambda}{\tau_R} = \frac{\int_0^l \kappa_\lambda \rho dl}{\int_0^l \kappa_R \rho dl}$$

$$\tau_\lambda = \int_0^{\tau_R} \frac{\kappa_\lambda}{\kappa_R} d\tau_R$$

Next

- Use the Saha and Boltzmann Equations to get populations
- You now have τ_λ
- Now compute τ_l by first computing α_l and using (a,u,Oν_0) as defined before. Note we are using wavelength as our variable.

$$\kappa_l(\tau_R) = N(\tau_R)\frac{\pi e^2}{m_e c}\frac{f}{\Delta\nu_0(\tau_R)}H(a,u)\left[1 - e^{\frac{-hc}{\lambda kT}}\right]$$

$$\tau_l = \int_0^l \kappa_l \rho \, dl$$