The Gray Case

Let us assume that the opacity is independent of frequency:

$$K_\nu \rightarrow K$$

NB: This does not mean $I_\nu = I$, that is, the intensity (flux, etc) ARE frequency dependent.

Equation of Transfer

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$

$$I \equiv \int_0^\infty I_\nu d\nu$$

$$S \equiv \int_0^\infty S_\nu d\nu$$

$$\therefore$$

$$\mu \frac{dI}{d\tau} = I - S$$

Assume Radiative equilibrium

$$\int_0^\infty \mathrm{K} J_\nu d\nu = \int_0^\infty \mathrm{K} S_\nu d\nu$$

$$\therefore$$

$$S = J$$

Continuing:

$$\mu \frac{dI}{d\tau} = I - J$$

$$J(\tau) = 1/2 \int_0^\infty J(\tau) E_1 \,|\, t - \tau \,|\, dt$$

- Assume LTE ==> $S_\nu = B_\nu$ ==> $J(\tau) = S(\tau) = B(\tau)$

  But $B(\tau) = \sigma T^4/\pi$

- Remember:

$$J = 1/4\pi \int I(\theta) d\omega$$

$$H = 1/4\pi \int I(\theta) \cos(\theta) d\omega$$

$$K = 1/4\pi \int I(\theta) \cos^2(\theta) d\omega$$

On We Go...

$$\cos\theta \frac{dI}{d\tau} = I - B$$

$$\cos\theta \frac{dI}{d\tau}\frac{d\omega}{4\pi} = I\frac{d\omega}{4\pi} - B\frac{d\omega}{4\pi}$$

$$\frac{1}{4\pi}\frac{d}{d\tau}(I\cos\theta d\omega) = \frac{1}{4\pi}Id\omega - \frac{1}{4\pi}Bd\omega$$

Now Integrate over all solid angles:

$$\frac{d}{d\tau}\frac{1}{4\pi}\int I\cos\theta d\omega = \frac{1}{4\pi}\int Id\omega - B$$

$$\frac{dH}{d\tau} = J - B = 0$$

$$\therefore$$

$$H = \textbf{Constant}$$

More Equations!

$$\mu \frac{dI}{d\tau} = I - J$$

Multiply by μdω/4π (i.e.. Take the 1st moment)

$$d\omega \frac{\mu^2}{4\pi} \frac{d}{d\tau} I = \frac{\mu}{4\pi}\left(I - J\right) d\omega$$

$$\frac{d}{d\tau} \frac{1}{4\pi} \int I \mu^2 d\omega = \frac{1}{4\pi} \int I \mu d\omega - \frac{1}{4\pi} \int J \mu d\omega$$

Note:

$$\frac{1}{4\pi} \int J \mu d\omega = \frac{1}{4\pi} \int B \mu d\omega$$

$$= \frac{B}{2} \int_{-1}^{1} \mu d\mu$$

$$= 0$$

Therefore:

$$dK / d\tau = H$$

$$K = H\tau + C$$

Make An Assumption!

- To proceed further (to this point we have been rather rigorous (e.g., we assume LTE and radiative equilibrium but do everything else properly).
- In the interior of a star I ≠ f(θ) so consider $K = 1/4\pi \int I(\theta) \cos^2\theta d\omega$
- But if I ≠ f(θ) then

$$K \approx \frac{I}{4\pi} 2\pi \left(\int_{-1}^{1} \mu^2 d\mu \right)$$

$$\approx \frac{I}{2} \left(\int_{-1}^{1} \mu^2 d\mu \right)$$

$$\approx \frac{I}{3}$$

What About J?

If $I \neq f(\theta)$ then $J = I/4\pi\int d\omega = I$ but this means

$$K \rightarrow J/3$$

Here's the approximation: $K = J/3$ everywhere. This is called the Eddington Approximation.

$$dK/d\tau = H$$

$$dJ/d\tau = 3H$$

$$\text{or } J = 3H\tau + C$$

Now Let us Consider the Intensities

Gray Case Intensity

We will consider that $I \neq f(\theta)$ except for the general distinction between an inward and an outward flow:

$$
\begin{aligned}
I(\theta) &= I_1 : \quad 0 < \theta < \pi/2 \quad \text{Outward} \\
&= I_2 : \quad \pi/2 < \theta < \pi \quad \text{Inward}
\end{aligned}
$$

We shall also assume that at the surface that I is constant over the hemisphere and that there is no inward flux ($I_2 = 0$) so

$$
\begin{aligned}
J &= \frac{1}{4\pi}\oint I d\omega \\
&= \frac{1}{4\pi}2\pi\left[\int_0^{\frac{\pi}{2}} I_1 \sin\theta d\theta + \int_{\frac{\pi}{2}}^{\pi} I_2 \sin\theta d\theta\right] \\
&= \frac{1}{2}\left(I_1 + I_2\right)
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2}\int I \cos\theta \sin\theta d\theta \\
&= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} I_1 \cos\theta \sin\theta d\theta + \int_{\frac{\pi}{2}}^{\pi} I_2 \cos\theta \sin\theta d\theta\right] \\
&= \frac{1}{4}\left(I_1 - I_2\right)
\end{aligned}
$$

Continuing On ....

What we want is the constant is the constant in

$J = 3H\tau + C$

At the Surface $I_2 = 0$ so

$$J_0 = I_0/2$$

$$H_0 = I_0/4$$

which means $H_0 = J_0/2$

So $J = 3H\tau + C$ evaluated at $\tau=0$ (the surface) yields:

$$2H = C$$

$$\therefore$$

$$J(\tau) = 3H\tau + 2H = H(3\tau + 2)$$

But note that $F^A = 4H$ so

$$J(\tau) = (F/4)(3\tau + 2)$$

The T - τ Relation

A Relation of Fundamental Importance is the relation between the temperature T and the optical depth τ.

For the Gray Case: $J(\tau) = H(3\tau + 2)$

But in LTE: $J = B$

$$\therefore$$

$$B(T) = H(3\tau + 2)$$

But this is the Integrated $B = \sigma T^4/\pi$ so

$$\sigma T^4/\pi = H(3\tau + 2)$$

More on T-τ

At $\tau_0 = 0$ we have $\sigma T_0^4/\pi = 2H$

so

$$\sigma T^4/\pi = 3H\tau + 2H = \sigma T_0^4/\pi + 3/2\ \sigma T_0^4/\pi\ \tau$$

$$T^4 = T_0^4 + 3/2\ T_0^4\ \tau$$

$$= T_0^4(1 + 3/2\ \tau)$$

But what is T_0? Well remember that $\mathbf{F^P} = 4\pi H = \sigma T^4$

and $F^A = 4H = \sigma T^4/\pi$

The definition of T_{eff} is $F^P = \sigma T_{eff}^4$

$$2H = \sigma T_{eff}^4/2\pi = \sigma T_0^4/\pi$$

The T-τ Relation

$$T_{eff} = \sqrt[4]{2} T_0$$

$$T_{eff} = 1.189 T_0$$

For the Sun T_{eff} = 5740 K which yields T_0 = 4826 K

so to specify the temperature structure

$T^4 = 1/2\ T_{eff}^4(1 + 3/2\ \tau)$

NB: $T = T_{eff}$ at $\tau = 2/3$