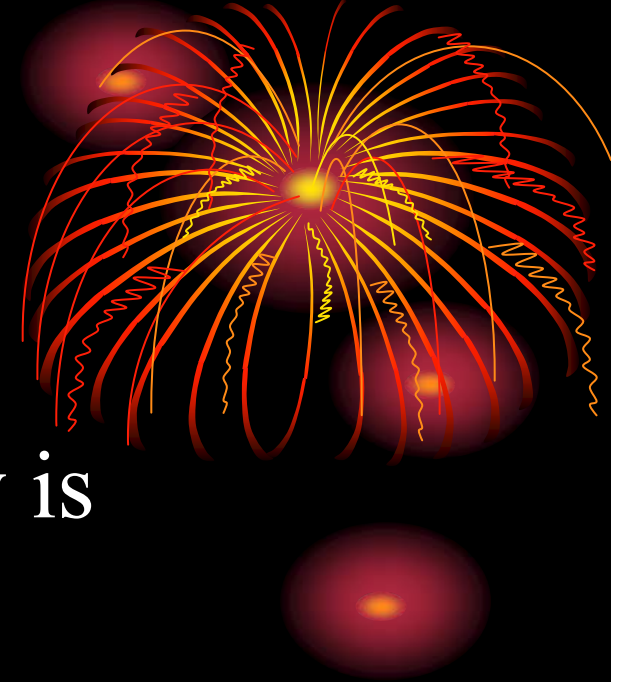


The Gray Case

Let us assume that the opacity is independent of frequency:

$$K_\nu \rightarrow K$$

NB: This does not mean $I_\nu = I$, that is, the intensity (flux, etc) ARE frequency dependent.



Equation of Transfer

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$

$$I \equiv \int_0^\infty I_\nu d\nu$$

$$S \equiv \int_0^\infty S_\nu d\nu$$

\therefore

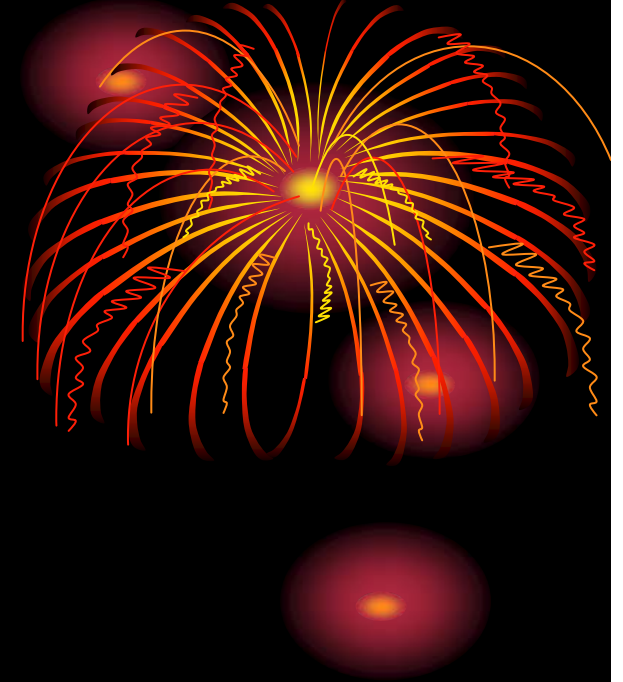
$$\mu \frac{dI}{d\tau} = I - S$$

Assume Radiative equilibrium

$$\int_0^\infty KJ_\nu d\nu = \int_0^\infty KS_\nu d\nu$$

\therefore

$$S = J$$



Continuing:

$$\mu \frac{dI}{d\tau} = I - J$$

$$J(\tau) = 1/2 \int_0^\infty J(\tau) E_1 |t - \tau| dt$$

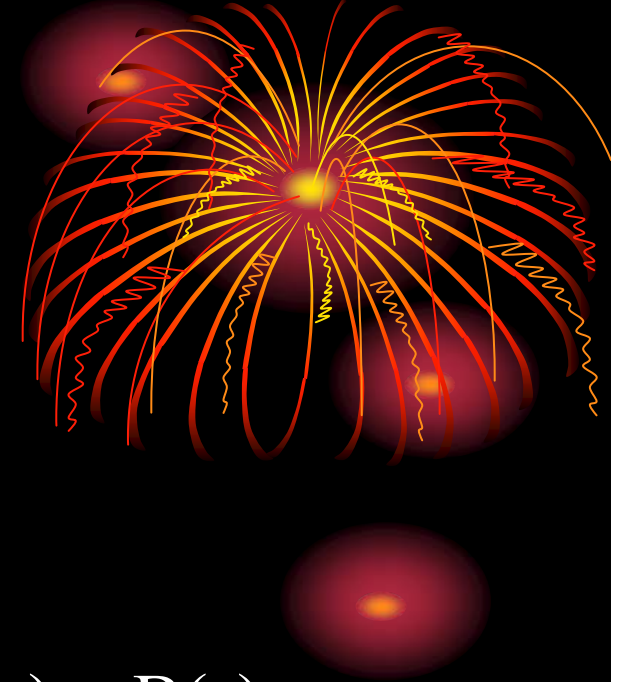
- Assume LTE $\implies S_\nu = B_\nu \implies J(\tau) = S(\tau) = B(\tau)$
But $B(\tau) = \sigma T^4 / \pi$

- Remember:

$$J = 1/4\pi \int I(\theta) d\omega$$

$$H = 1/4\pi \int I(\theta) \cos(\theta) d\omega$$

$$K = 1/4\pi \int I(\theta) \cos^2(\theta) d\omega$$



On We Go...

$$\cos \theta \frac{dI}{d\tau} = I - B$$

$$\cos \theta \frac{dI}{d\tau} \frac{d\omega}{4\pi} = I \frac{d\omega}{4\pi} - B \frac{d\omega}{4\pi}$$

$$\frac{1}{4\pi} \frac{d}{d\tau} (I \cos \theta d\omega) = \frac{1}{4\pi} I d\omega - \frac{1}{4\pi} B d\omega$$

Now Integrate over all solid angles:

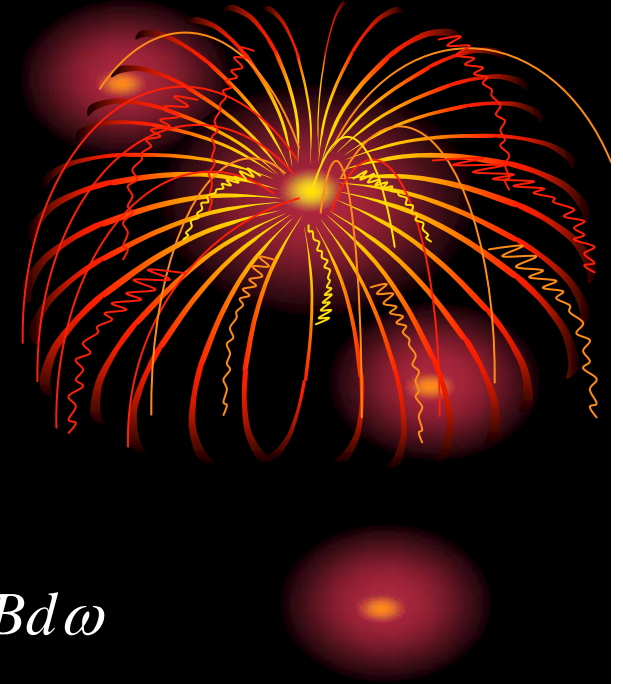
$$\frac{d}{d\tau} \frac{1}{4\pi} \int I \cos \theta d\omega = \frac{1}{4\pi} \int I d\omega - B$$

$$\frac{dH}{d\tau} = J - B = 0$$

\therefore

$$H = \text{Constant}$$

Gray Case



More Equations!

$$\mu \frac{dI}{d\tau} = I - J$$

Multiply by $\mu d\omega/4\pi$ (i.e.. Take the 1st moment)

$$d\omega \frac{\mu^2}{4\pi} \frac{d}{d\tau} I = \frac{\mu}{4\pi} (I - J) d\omega$$

$$\frac{d}{d\tau} \frac{1}{4\pi} \int I \mu^2 d\omega = \frac{1}{4\pi} \int I \mu d\omega - \frac{1}{4\pi} \int J \mu d\omega$$

Note:

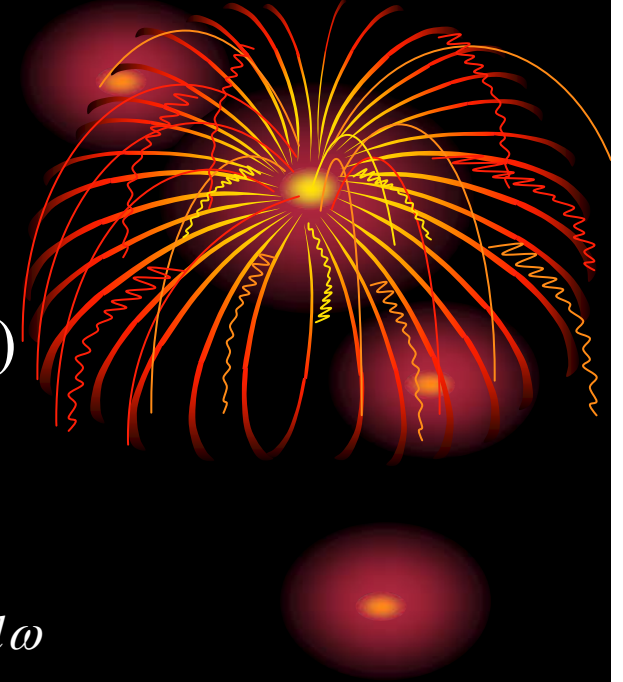
$$\begin{aligned} \frac{1}{4\pi} \int J \mu d\omega &= \frac{1}{4\pi} \int B \mu d\omega \\ &= \frac{B}{2} \int_{-1}^1 \mu d\mu \\ &= 0 \end{aligned}$$

Therefore:

$$dK / d\tau = H$$

$$K = H\tau + C$$

Gray Case



Make An Assumption!



- To proceed further (to this point we have been rather rigorous (e.g., we assume LTE and radiative equilibrium but do everything else properly).
- In the interior of a star $I \neq f(\theta)$ so consider $K = 1/4\pi \int I(\theta) \cos^2\theta d\omega$
- But if $I \neq f(\theta)$ then

$$K \approx \frac{I}{4\pi} 2\pi \left(\int_{-1}^1 \mu^2 d\mu \right)$$

$$\approx \frac{I}{2} \left(\int_{-1}^1 \mu^2 d\mu \right)$$

$$\approx \frac{I}{3}$$

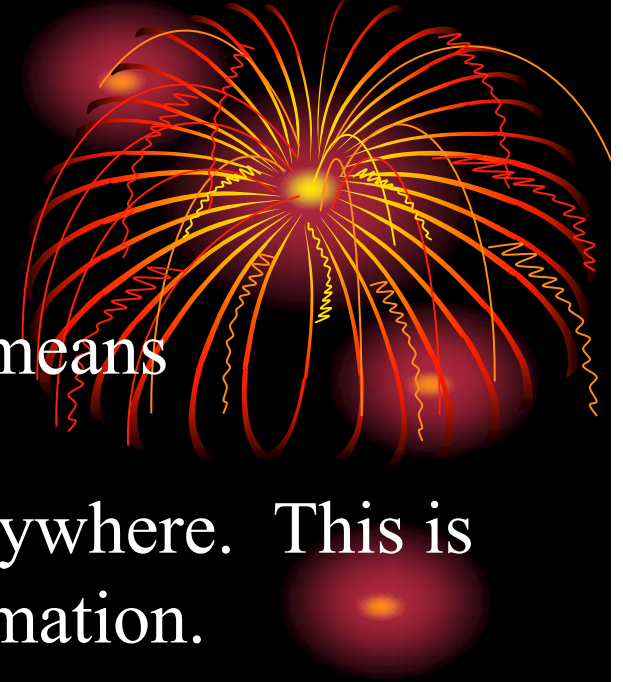
What About J?

If $I \neq f(\theta)$ then $J = I/4\pi \int d\omega = I$ but this means
 $K \rightarrow J/3$

Here's the approximation: $K = J/3$ everywhere. This is called the Eddington Approximation.

$$\begin{aligned}dK/d\tau &= H \\dJ/d\tau &= 3H \\ \text{or } J &= 3H\tau + C\end{aligned}$$

Now Let us Consider the Intensities



Gray Case Intensity

We will consider that $I \neq f(\theta)$ except for the general distinction between an inward and an outward flow:

$$\begin{aligned} I(\theta) &= I_1 : & 0 < \theta < \pi/2 & \text{Outward} \\ &= I_2 : & \pi/2 < \theta < \pi & \text{Inward} \end{aligned}$$

We shall also assume that at the surface that I is constant over the hemisphere and that there is no inward flux ($I_2 = 0$) so

$$\begin{aligned} J &= \frac{1}{4\pi} \oint I d\omega \\ &= \frac{1}{4\pi} 2\pi \left[\int_0^{\pi/2} I_1 \sin \theta d\theta + \int_{\pi/2}^{\pi} I_2 \sin \theta d\theta \right] \\ &= \frac{1}{2} (I_1 + I_2) \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2} \int I \cos \theta \sin \theta d\theta \\ &= \frac{1}{2} \left[\int_0^{\pi/2} I_1 \cos \theta \sin \theta d\theta + \int_{\pi/2}^{\pi} I_2 \cos \theta \sin \theta d\theta \right] \\ &= \frac{1}{4} (I_1 - I_2) \end{aligned}$$

Gray Case



Continuing On

What we want is the constant is the constant in
 $J = 3H\tau + C$

At the Surface $I_2 = 0$ so

$$J_0 = I_0/2$$

$$H_0 = I_0/4$$

which means $H_0 = J_0/2$

So $J = 3H\tau + C$ evaluated at $\tau=0$ (the surface) yields:

$$2H = C$$

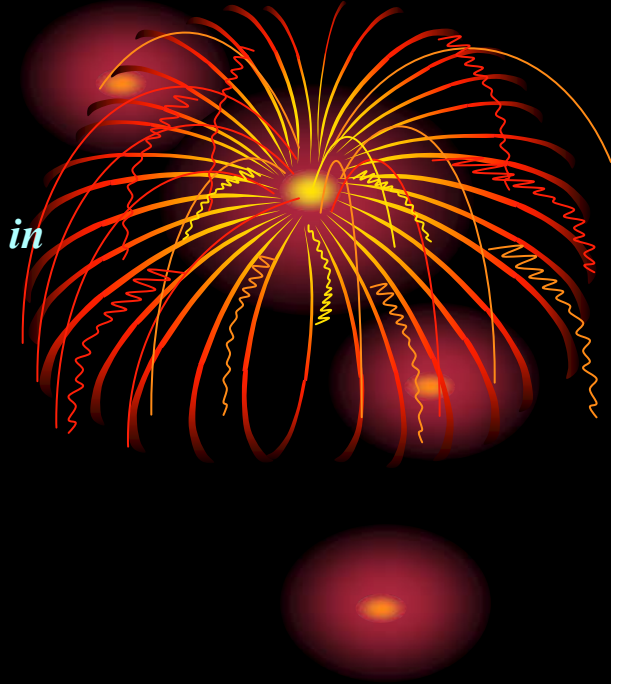
\therefore

$$J(\tau) = 3H\tau + 2H = H(3\tau + 2)$$

But note that $F^A = 4H$ so

$$J(\tau) = (F/4)(3\tau + 2)$$

Gray Case



The T - τ Relation

A Relation of Fundamental Importance is the relation between the temperature T and the optical depth τ .

For the Gray Case: $J(\tau) = H(3\tau + 2)$

But in LTE: $J = B$

\therefore

$$B(T) = H(3\tau + 2)$$

But this is the Integrated $B = \sigma T^4/\pi$ so

$$\sigma T^4/\pi = H(3\tau + 2)$$

Gray Case

More on T- τ

At $\tau_0 = 0$ we have $\sigma T_0^4/\pi = 2H$

so

$$\sigma T^4/\pi = 3H\tau + 2H = \sigma T_0^4/\pi + 3/2 \sigma T_0^4/\pi \tau$$

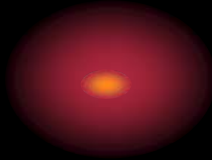
$$\begin{aligned} T^4 &= T_0^4 + 3/2 T_0^4 \tau \\ &= T_0^4(1 + 3/2 \tau) \end{aligned}$$

But what is T_0 ? Well remember that $F^P = 4\pi H = \sigma T^4$

$$\text{and } F^A = 4H = \sigma T^4/\pi$$

The definition of T_{eff} is $F^P = \sigma T_{\text{eff}}^4$

$$2H = \sigma T_{\text{eff}}^4/2\pi = \sigma T_0^4/\pi$$



The T- τ Relation

$$T_{eff} = \sqrt[4]{2} T_0$$

$$T_{eff} = 1.189 T_0$$

For the Sun $T_{eff} = 5740$ K which yields $T_0 = 4826$ K
so to specify the temperature structure

$$T^4 = 1/2 T_{eff}^4 (1 + 3/2 \tau)$$

$$\text{NB: } T = T_{eff} \text{ at } \tau = 2/3$$

