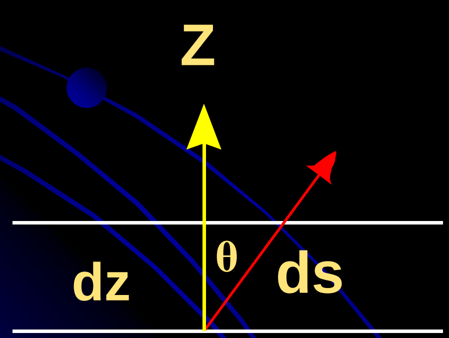
The Equation of Transfer

Basic Ideas

- Change = Source - Sink
  - Change In Intensity = Emitted Energy -“Absorbed” Energy
- $dI_\nu d\nu d\mu dt dA = j_\nu(\rho dA ds) d\nu d\mu dt - K_\nu I_\nu(\rho dA ds) d\nu d\mu dt$
- $dI_\nu = \rho j_\nu ds - \rho K_\nu I_\nu ds$



$$dI_\nu = j_\nu(\rho/\mu)\, dz - K_\nu\, I_\nu\, (\rho/\mu)\, dz$$

$$\mu\, dI\nu / \rho dz = j_\nu - K_\nu I_\nu$$

Remember: $d\tau_\nu = -\rho K_\nu dz$

$$\mu(dI_\nu/d\tau_\nu) = I_\nu - S_\nu$$

Where $S_\nu \equiv j_\nu / K_\nu$

The geometry gives: $dz = \mu ds$

The Source Function S_ν

- K_ν is the total absorption coefficient
  - $K_\nu = \kappa_\nu + \sigma_\nu$
- The equation of transfer is $\mu(dI_\nu/d\tau_\nu) = I_\nu - S_\nu$
- The Emission Coefficient $j_\nu = j_\nu^t + j_\nu^s$
  - If there is no scattering: $j_\nu^s = 0$ and $\sigma_\nu = 0$
  - Then $j_\nu = j_\nu^t$ and $K_\nu = \kappa_\nu$
- Thus $S_\nu = j_\nu / K_\nu = j_\nu^t / \kappa_\nu = B_\nu(T)$
- In the case of no scattering the source function is the Planck Function!

The Slab Redux

- Slab: No Emission but absorption
- $\mu dI_\nu/\rho dz = - K_\nu I_\nu$
- $\mu dI = -\rho K_\nu dz\ I_\nu$
- $dI_\nu/I_\nu = d\tau_\nu/\mu$
- $I(0,\mu) = I(\tau_\nu,\mu)e^{-\tau/\mu}$
- Note that the previous solution was for $\mu = 1$ which is $\theta = 0$.

Formal Solution

- $\mu dI/d\tau = I - S$: Suppress the frequency dependence
- It is a linear first-order differential equation with constant coefficients and thus must have an integrating factor: $e^{-\tau/\mu}$
- $d/d\tau\,(Ie^{-\tau/\mu}) = -S\,e^{-\tau/\mu} / \mu$
- $dI/d\tau\,e^{-\tau/\mu} + I\,e^{-\tau/\mu}\,(-1/\mu) = -S\,e^{-\tau/\mu} / \mu$
- $dI/d\tau + I/-\mu = -S/\mu$
- $\mu dI/d\tau = -S + I$

$$Ie^{\frac{-\tau}{\mu}}\Big|_{\tau_1}^{\tau_2} = -\int_{\tau_1}^{\tau_2} S(t)e^{\frac{-t}{\mu}}dt/\mu$$

so

$$I(\tau_1,\mu) = I(\tau_2,\mu)e^{\frac{-(\tau_2-\tau_1)}{\mu}} + \int_{\tau_1}^{\tau_2} S(t)e^{\frac{-(t-\tau_1)}{\mu}}dt/\mu$$

Boundary Conditions

- For the optical depths: $\tau_1 = 0$ and $\tau_2 \rightarrow \infty$
- We Require: $\lim_{\tau \to \infty} I_\nu(\tau_\nu, \mu) e^{\frac{-\tau_\nu}{\mu}} = 0$
- The above is the lower boundary "boundedness" condition. The upper boundary condition is that there be no incoming radiation.

$$I(0, \mu) = \int_0^\infty S(t)\, e^{\frac{-t}{\mu}} \frac{dt}{\mu}$$

Physical Meaning

$$I(0,\mu) = \int_0^\infty S(t)\, e^{\frac{-t}{\mu}} \frac{dt}{\mu}$$

- The physical meaning of the solution is that the emergent intensity is the weighted mean of the source function with the weights proportional to the fraction of energy getting to the surface at each wavelength / frequency. The solution depends on the form of S:
- Let $S = a+bt$
- Then $I(0,\mu) = a + b\mu$

Separate τ Into Domains

Incoming and Outgoing at an Arbitrary Depth

"Outgoing"

$$I(\tau,\mu)=\int_{\tau}^{\infty}S(t)e^{\frac{-(t-\tau)}{\mu}}\frac{dt}{\mu} \qquad 0\leq\mu\leq 1 \;\; or \;\; 90\geq\theta\geq 0$$

"Incoming"

$$I(\tau,\mu)=\int_{\tau}^{0}S(t)e^{\frac{-(t-\tau)}{\mu}}\frac{dt}{\mu} \qquad -1\leq\mu\leq 0 \;\; or \;\; 180\geq\theta\geq 90$$

What Is the Difficulty?

These appear simple enough but …

- Consider the Source Function S
- What is the "Usual" form for S?
  - For coherent isotropic scattering (a very common case):
  - $S_\nu = (\kappa_\nu/(\kappa_\nu + \sigma_\nu))\, B_\nu + (\sigma_\nu /(\kappa_\nu + \sigma_\nu))\, J_\nu$
- So the solution of the equation of transfer is:

$$I(0,\mu) = \int_0^\infty S(t)\, e^{\frac{-t}{\mu}} \frac{dt}{\mu}$$

- But J depends on I: $J_\nu = (1/4\pi)\int I_\nu d\mu$
- Therefore S depends on I
- This is not an easy problem in general!

Schwarzschild-Milne Equations

- Let us look at the Mean Intensity J_ν

$$J(\tau) \equiv 1/2\int_{-1}^{1} I(\tau,\mu)d\mu$$

$$= 1/2\int_{\tau}^{\infty} S(t)\int_{0}^{1} e^{\frac{-(t-\tau)}{\mu}} \frac{d\mu}{\mu} dt + 1/2\int_{0}^{\tau} S(t)\int_{-1}^{0} e^{\frac{-(\tau-t)}{-\mu}} \frac{d\mu}{-\mu} dt$$

- In the second integral we have 1) reversed the order of integration so μ => -μ and 2) changed the sign on both parts of the exponent. (See Slide 8).

Continuing ...

$$J(\tau) = 1/2\int_{\tau}^{\infty} S(t)\int_{1}^{\infty} e^{-w(t-\tau)}\frac{dw}{w}dt + 1/2\int_{0}^{\tau} S(t)\int_{1}^{\infty} e^{-w(\tau-t)}\frac{dw}{w}dt$$

- For integrand 1: w = μ^{-1} => dw = $-\mu^{-2}d\mu$ so dw/w = -dμ/μ. For the second w = $-\mu^{-1}$ for which dw/w = -dμ/μ also. In the first integral we also changed the order of integration (which cancelled the minus sign on dw/w).

$$J(\tau) = 1/2\int_{0}^{\infty} S(t)\left[\int_{1}^{\infty} e^{-w|t-\tau|}\frac{dw}{w}\right]dt$$

The Exponential Integral

The Integrals in J(τ) are Called the Exponential Integrals

$$E_n(x) \equiv \int_1^\infty \frac{e^{-xt}}{t^n} dt$$

$$= x^{n-1} \int_x^\infty \frac{e^{-t}}{t^n} dt$$

These integrals are recursive!

Schwarzschild-Milne Equations

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)E_1(|\,t_\nu - \tau_\nu\,|)dt_\nu$$

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)E_2(|\,t_\nu - \tau_\nu\,|)dt_\nu$$

The Mean Intensity Equation is called the Schwarzschild Equation and the flux equation is called the Milne equation. The surface flux is:

$$H_\nu(0) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)E_2(t_\nu)dt_\nu$$