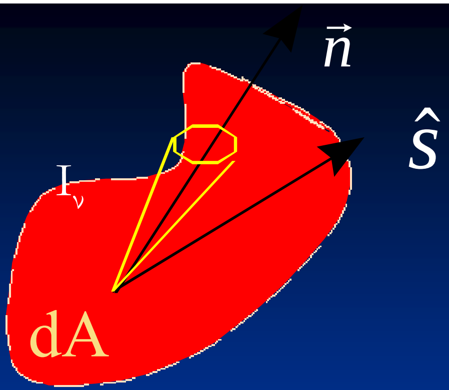
Basic Definitions

Terms we will need to discuss radiative transfer.



Specific Intensity I

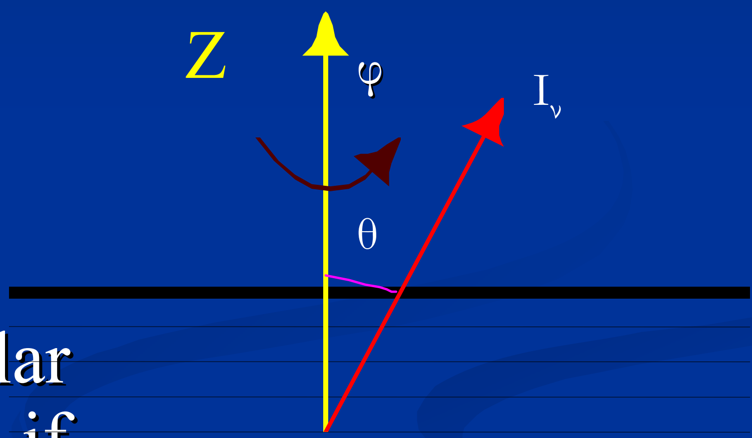
- I_ν(**r**,**n**,t) ≡ the amount of energy at position **r** (vector), traveling in direction **n** (vector) at time t per unit frequency interval, passing through a unit area oriented normal to the beam, into a unit solid angle in a second (unit time).
- If θ is the angle between the normal **s** (unit vector) of the reference surface dA and the direction **n** then the energy passing through dA is
- $$dE_\nu = I_\nu(\mathbf{r},\mathbf{n},t)\, dA \cos(\theta)\, d\nu\, d\omega\, dt$$
- I_ν is measured in erg Hz^{-1} s^{-1} cm^{-2} $steradian^{-1}$

Specific Intensity II

- We shall only consider time independent properties of radiation transfer
  - Drop the t
- We shall restrict ourselves to the case of plane-parallel geometry.
  - Why: the point of interest is d/D where d = depth of atmosphere and D = radius of the star.
  - The formal requirement for plane-parallel geometry is that d/R ~ 0
  - For the Sun: d ~ 500 km, D ~ $7(10^5)$ km
    - d/D ~ $7(10^{-4})$ for the Sun
    - The above ratio is typical for dwarfs, supergiants can be of order 0.3.

Specific Intensity III

- Go to the geometric description (z,θ,φ) - polar coordinates
  - θ is the polar angle
  - φ is the azimuthal angle
  - Z is with respect to the stellar boundary (an arbitrary idea if there ever was one).
    - Z is measured positive upwards
      - \+ above "surface"
      - \- below "surface"



Mean Intensity: $J_\nu(z)$

- Simple Average of I over all solid angles

$$J_\nu = \oint I_\nu(z,\theta,\phi) d\omega / \oint d\omega$$

but

$$\oint d\omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta d\theta d\phi = 4\pi$$

Therefore

$$J_\nu = (1/4\pi) \int_0^{2\pi} d\phi \int_0^{\pi} I_\nu(z,\theta,\phi) \sin\theta d\theta$$

- I_ν is generally considered to be independent of φ so

$$J_\nu = 1/2 \int_0^{\pi} I_\nu(z,\theta) \sin\theta d\theta$$

$$= 1/2 \int_{-1}^{1} I_\nu(z,\mu) d\mu$$

- Where μ = cosθ ==> dμ = -sinθdθ
- The lack of azimuthal dependence implies homogeneity in the atmosphere.

Physical Flux

- Flux ≡ Net rate of Energy Flow across a unit area. It is a vector quantity.

$$F_\nu^P = \oint I_\nu(\vec{r},\vec{n})\vec{n}d\omega$$

- Go to (z, θ, φ) and ask for the flux through one of the plane surfaces:

$$F_\nu^P = F_\nu^P \cdot \hat{k} = \oint I_\nu(z,\theta,\phi)\cos\theta d\omega$$

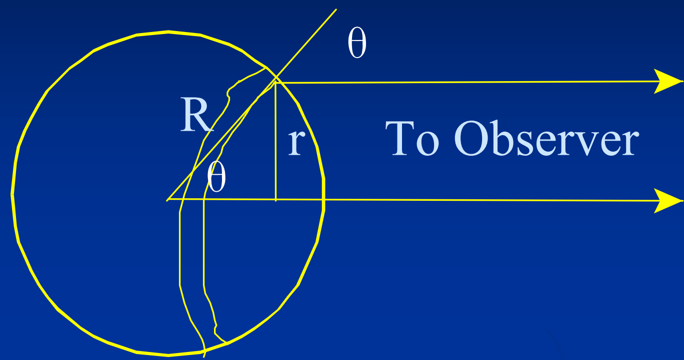
$$= 2\pi\int_{-1}^{1} I_\nu(z,\mu)\mu d\mu$$

- Note that the azimuthal dependence has been dropped!

Astrophysical Flux

$$F_\nu^A = (1/\pi) F_\nu^P$$

$$= 2\int_{-1}^{1} I_\nu(z,\mu)\mu d\mu$$



- F_ν^P is related to the observed flux!
  - R = Radius of star
  - D = Distance to Star
  - D>>R ➔ All rays are parallel at the observer
- Flux received by an observer is $df_\nu = I_\nu\, d\omega$ where
  - dω = solid angle subtended by a differential area on stellar surface
  - I_ν = Specific intensity of that area.

Astrophysical Flux II

For this Geometry:

$$dA = 2\pi r dr$$

but $R \sin\theta = r$

$$dr = R \cos\theta \, d\theta$$

$$dA = 2R \sin\theta \, R\cos\theta \, d\theta$$

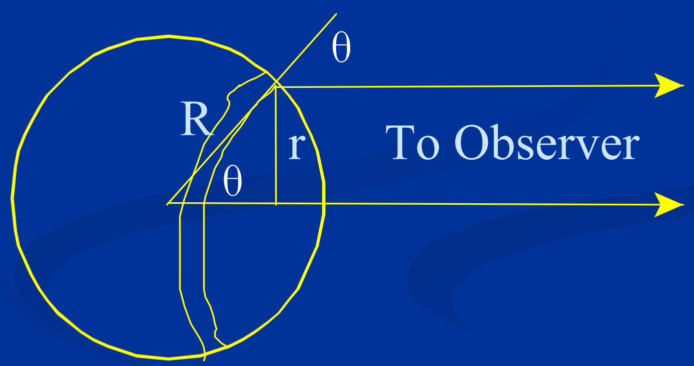
$$dA = 2R^2 \sin\theta \cos\theta \, d\theta$$

Now $\mu = \cos\theta$

$$dA = 2R^2 \mu d\mu$$

By definition $d\omega = dA/D^2$

$$d\omega = 2(R/D)^2 \mu \, d\mu$$



Astrophysical Flux III

Now from the annulus the radiation emerges at angle θ so the appropriate value of I_ν is $I_\nu(0,\mu)$. Now integrate over the star: Remember $df_\nu = I_\nu d\omega$

$$f_\nu = 2\pi(\frac{R}{D})^2 \int_0^1 I_\nu(0,\mu)\mu d\mu$$

$$= (\frac{R}{D})^2 F_\nu^P = (\frac{R}{D})^2 \pi F_\nu^A$$

NB: We have assumed $I(0,-\mu) = 0$ ==> No incident radiation. We observe f_ν for stars: not F_ν^P

Inward μ: $-1 < \mu < 0$ Outward μ: $0 < \mu < -1$

Moments of the Radiation Field

$$M_\nu(z,n) = 1/2\int_{-1}^{1} I_\nu(z,\mu)\mu^n d\mu$$

Order 0 : (the Mean Intensity)

$$J_\nu(z) = 1/2\int_{-1}^{1} I_\nu(z,\mu)d\mu$$

Order 1 : (the Eddington Flux)

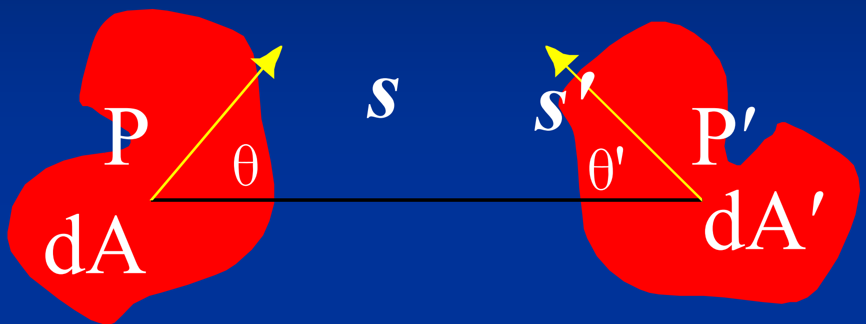
$$H_\nu(z) = 1/2\int_{-1}^{1} I_\nu(z,\mu)\mu d\mu$$

Order 1 : (the K Integral)

$$K_\nu(z) = 1/2\int_{-1}^{1} I_\nu(z,\mu)\mu^2 d\mu$$

Invariance of the Specific Intensity

- The definition of the specific intensity leads to invariance.



Consider the ray packet which goes through dA at P and dA' at P'

$dE_\nu = I_\nu dA\cos\theta d\omega d\nu dt = I_\nu' dA' \cos\theta' d\omega' d\nu dt$

- $d\omega$ = solid angle subtended by dA' at P = $dA'\cos\theta'/r^2$
- $d\omega'$ = solid angle subtended by dA at P' = $dA\cos\theta/r^2$

- $dE_\nu = I_\nu dA\cos\theta\, dA'\cos\theta'/r^2\, d\nu dt = I_\nu' dA'\cos\theta' dA\cos\theta/r^2\, d\nu dt$
- This means $I_\nu = I_\nu'$
- Note that dE_ν contains the inverse square law.

Energy Density I

- Consider an infinitesimal volume V into which energy flows from all solid angles. From a specific solid angle $d\omega$ the energy flow through an element of area dA is
- $\delta E = I_\nu dA\cos\theta d\omega d\nu dt$
- Consider only those photons in flight across V. If their path length while in V is ℓ, then the time in V is $dt = \ell/c$. The volume they sweep is $dV = \ell dA\cos\theta$. Put these into δE_ν:
- $\delta E_\nu = I_\nu (dV/\ell\cos\theta) \cos\theta\, d\omega\, d\nu\, \ell/c$
- $\delta E_\nu = (1/c)\, I_\nu\, d\omega d\nu dV$

Energy Density II

- Now integrate over volume
  - $E_\nu d\nu = [(1/c)\int_V dV \oint I_\nu d\omega] d\nu$
- Let $V \rightarrow 0$: then I_ν can be assumed to be independent of position in V.
- Define Energy Density $U_\nu \equiv E_\nu / V$
- $E_\nu = (V/c) \oint I_\nu d\omega$
- $U_\nu = (1/c) \oint I_\nu d\omega = (4\pi/c) J_\nu$
  - $J_\nu = (1/4\pi) \oint I_\nu d\omega$ by definition

Photon Momentum Transfer

- Momentum Per Photon = mc = mc^2/c = $h\nu/c$
  - Mass of a Photon = $h\nu/c^2$
- Momentum of a pencil of radiation with energy $dE_\nu = dE_\nu/c$
- $dp_r(\nu) = (1/dA)\,(dE_\nu \cos\theta/c)$

  $= ((I_\nu dA \cos\theta d\omega)\cos\theta) / c dA$

  $= I_\nu \cos^2\theta d\omega/c$
- Now integrate: $p_r(\nu) = (1/c) \int I_\nu \cos^2\theta d\omega$

  $= (4\pi/c) K_\nu$

Radiation Pressure

Photon Momentum Transport Redux

- It is the momentum rate per unit and per unit solid angle = Photon Flux * ("m"v per photon) * Projection Factor
- $dp_r(\nu) = (dE_\nu/(h\nu dtdA)) * (h\nu/c) * \cos\theta$ where $dE_\nu = I_\nu d\nu dA\cos\theta d\omega dt$ so $dp_r(\nu) = (1/c)\, I_\nu \cos^2\theta d\omega d\nu$
- Integrate $dp_r(\nu)$ over frequency and solid angle:

$$p_r = \frac{1}{c}\int_0^\infty \int_{4\pi} I_\nu \cos^2\theta d\omega d\nu$$

Or in terms of frequency

$$p_r(\nu) = \frac{1}{c}\int_{4\pi} I_\nu \cos^2\theta d\omega$$

$$= \frac{4\pi}{c} K_\nu$$

Isotropic Radiation Field

$$I(\mu) \neq f(\mu)$$

$$\begin{aligned}
J_\nu(z) &= 1/2\int_{-1}^{1} I_\nu(z)\,d\mu \\
&= 1/2 I_\nu(\mu|_{-1}^{1}) \\
&= 1/2 I_\nu(1-(-1)) \\
&= I_\nu
\end{aligned}$$

$$\begin{aligned}
K_\nu(z) &= 1/2\int_{-1}^{1} I_\nu \mu^2 d\mu \\
&= 1/2 I_\nu \int_{-1}^{1} \mu^2 d\mu \\
&= 1/2 I_\nu (\frac{\mu^3}{3}|_{-1}^{1}) \\
&= 1/2 I_\nu (1/3-(-1/3)) \\
&= I_\nu / 3 \\
&= J_\nu / 3
\end{aligned}$$