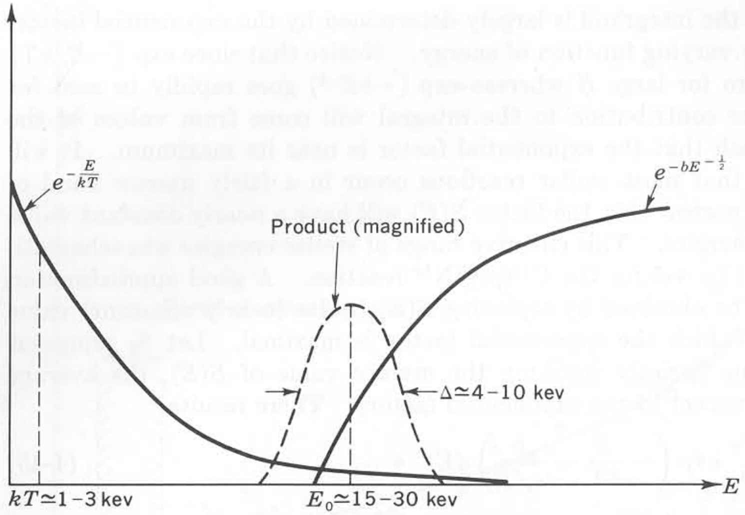

Fig. 4-6 The dominant energy-dependent factors in thermonuclear reactions. Most of the reactions occur in the high-energy tail of the maxwellian energy distribution, which introduces the rapidly falling factor $\exp(-E/kT)$. Penetration through the coulomb barrier introduces the factor $\exp(-bE^{-\frac{1}{2}})$, which vanishes strongly at low energy. Their product is a fairly sharp peak near an energy designated by E_0, which is generally much larger than kT. The peak is pushed out to this energy by the penetration factor, and it is therefore commonly called the *Gamow peak* in honor of the physicist who first studied the penetration through the coulomb barrier.

Standard techniques exist for determining the extent of the error made in the method of steepest descents. It will be more instructive to apply such analysis to the problem at hand, however, than to concern ourselves further with the general technique.

The integrand in Eq. (4-44) is a sharply peaked function, being the product of an exponential that vanishes at large energy, $\exp(-E/kT)$, and an exponential that vanishes at low energy, $\exp(-bE^{-\frac{1}{2}})$, as illustrated schematically in Fig. 4-6. All other things being equal, the particles that are most effective in causing nuclear reactions are those pairs having energies near E_0. The value of E_0 is determined from the location of the maximum of the integrand:

$$\frac{d}{dE}\left(\frac{E}{kT} + bE^{-\frac{1}{2}}\right)_{E=E_0} = \frac{1}{kT} - \frac{1}{2}bE_0^{-\frac{3}{2}} = 0$$

or

$$E_0 = \left(\frac{bkT}{2}\right)^{\frac{2}{3}} \tag{4-46}$$

Problem 4-9: Show that

$$E_0 = 1.220(Z_1^2 Z_2^2 A T_6^2)^{\frac{1}{3}} \quad \text{kev} \tag{4-47}$$

where T_6 is the temperature in millions of degrees. This energy is frequently called the *most effective energy for thermonuclear reactions.*

Evaluation of Eq. (4-47) shows that for normal light nuclei and temperatures of some tens of millions of degrees, the most effective energy E_0 is usually 10 to 30 kev. This energy is greater than $kT = 0.086T_6$ kev, reflecting the fact that the barrier-penetration factor has favored the selection of particles on the high-energy tail of the Maxwell-Boltzmann energy distribution.

The method of steepest descent is equivalent to the replacement of a sharply peaked exponential function by a gaussian function having a maximum of the same size and the same curvature at the maximum, in this case at $E = E_0$. That is, the integral will be evaluated by the replacement

$$\exp\left(-\frac{E}{kT} - bE^{-\frac{1}{2}}\right) \approx C \exp -\left(\frac{E - E_0}{\Delta/2}\right)^2 \tag{4-48}$$

where clearly

$$C = \exp\left(-\frac{E_0}{kT} - bE_0^{-\frac{1}{2}}\right) \tag{4-49}$$

and where the $1/e$ width, $\Delta/2$, is estimated by the requirement that the second derivatives match at E_0.

Problem 4-10: Show that the constant C is also equal to

$$C = \exp - \frac{3E_0}{kT} \tag{4-50}$$

and the full width at $1/e$ is

$$\Delta = \frac{4}{\sqrt{3}}(E_0 kT)^{\frac{1}{2}} \tag{4-51}$$

$$\Delta = 0.75(Z_1^2 Z_2^2 A T_6^5)^{\frac{1}{6}} \quad \text{kev} \tag{4-52}$$

It is apparent from Eq. (4-51) that the full width is approximately twice the geometric mean of kT and the peak energy E_0 but is still smaller than E_0 itself.

Problem 4-11: Show that, for the reaction $C^{12}(p,\gamma)N^{13}$,

$E_0 = 3.93T_6^{\frac{2}{3}}$ kev

$\Delta = 1.35T_6^{\frac{5}{6}}$ kev

and evaluate at the center of the sun, where $T = 15 \times 10^6$ °K. The numerical value of the exponential factor for $T_6 = 30$ is plotted in Fig. 4-7. The energies at $1/e$ of maximum are shown.