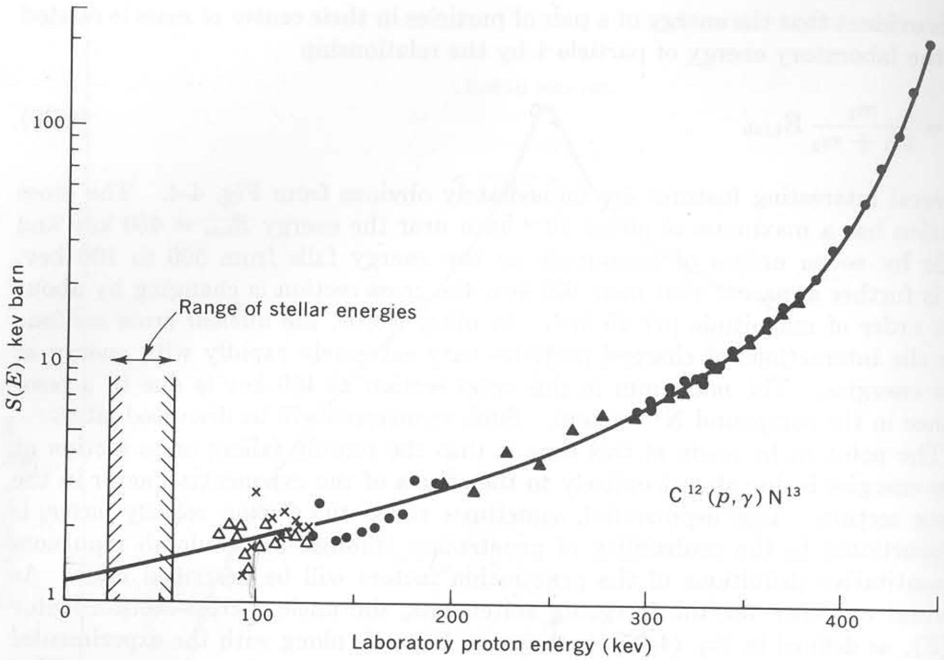

Fig. 4-5 The cross-section factor $S(E)$ for the radiative capture of protons by C^{12}. The differing types of data points represent five different experiments performed at different times and laboratories by the workers indicated. Detailed references and discussion may be found in D. F. Hebbard and J. L. Vogl, *Nucl. Phys.*, **21**:652 (1960). This curve is more readily extrapolated than the one in Fig. 4-4.

The velocity distribution may be written as the following normalized energy distribution:

$$\psi(E)\,dE = \phi(v)\,dv = -\frac{2}{\sqrt{\pi}}\frac{E}{kT}\exp\left(-\frac{E}{kT}\right)\frac{dE}{(kTE)^{\frac{1}{2}}} \tag{4-43}$$

In the nonresonant-reaction case, the cross-section factor $S(E)$ is slowly varying over the range of energies that are important in stellar interiors, and so in that case Eq. (4-37) may be a useful substitution for $\sigma(E)$ in the calculation of the reaction rate per pair of particles:

$$\lambda = \langle \sigma v \rangle = \int_0^\infty \sigma(E)v(E)\psi(E)\,dE$$

$$= \int_0^\infty \frac{S(E)}{E}\exp\left(-bE^{-\frac{1}{2}}\right)\sqrt{\frac{2E}{\mu}}\frac{2}{\sqrt{\pi}}\frac{E}{kT}\exp\left(-\frac{E}{kT}\right)\frac{dE}{(kTE)^{\frac{1}{2}}}$$

$$= \left(\frac{8}{\mu\pi}\right)^{\frac{1}{2}}\frac{1}{(kT)^{\frac{3}{2}}}\int_0^\infty S(E)\exp\left(-\frac{E}{kT}-bE^{-\frac{1}{2}}\right)dE \tag{4-44}$$

The behavior of the integrand is largely determined by the exponential factor, since it is a rapidly varying function of energy. Notice that since $\exp(-E/kT)$ goes rapidly to zero for large E whereas $\exp(-bE^{-\frac{1}{2}})$ goes rapidly to zero for small E, the major contribution to the integral will come from values of the energy that are such that the exponential factor is near its maximum. It will soon be apparent that most stellar reactions occur in a fairly narrow band of stellar energies, so narrow that the factor $S(E)$ will have a nearly constant value over the band of energies. This effective range of stellar energies was schematically indicated in Fig. 4-5 for the $C^{12}(p,\gamma)N^{13}$ reaction. A good approximation to Eq. (4-44) will be obtained by replacing $S(E)$ by its (nearly constant) value at the energy for which the exponential factor is maximal. Let S_0 represent that constant value [strictly speaking the *average* value of $S(E)$, the average being taken with respect to the exponential factor]. There results

$$\lambda = \left(\frac{8}{\mu\pi}\right)^{\frac{1}{2}}\frac{S_0}{(kT)^{\frac{3}{2}}}\int_0^\infty \exp\left(-\frac{E}{kT}-\frac{b}{\sqrt{E}}\right)dE \tag{4-45}$$

which can be evaluated by approximating the integrand by an appropriate gaussian.

Such a procedure is the simplest example of a method of doing a certain class of integrals, called the *method of steepest descent*. The method is applicable to integrals of the form

$$\int g(x)e^{-f(x)}\,dx$$

where $g(x)$ is a slowly varying function of x and the function $f(x)$ has a value much larger than unity and a single sharp minimum at x_0. In those circumstances, the integral may be approximated by expanding $f(x)$:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \cdots$$

$$\approx f(x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$$

since the first derivative vanishes at the minimum, and higher terms are discarded as being important only for those relatively large values of $x - x_0$ for which $f(x) \gg f(x_0)$, a fact necessitating that there be little contribution to the integral. Then a good estimate for the value of the integral becomes

$$g(x_0)e^{-f(x_0)}\int_{-\infty}^{\infty}\exp\left[-f''(x_0)\frac{(x-x_0)^2}{2}\right]dx$$

which has an elementary value.

Problem 4-8: Show that the integral is approximately

$$\int g(x)e^{-f(x)}\,dx \approx \sqrt{\frac{2\pi}{f''(x_0)}}\,g(x_0)\,e^{-f(x_0)}$$

Of course this approximation is useless unless $f(x)$ has the properties prescribed for it.