ORBITS & KEPLER'S LAWS

Kepler's laws: planets around Sun, moons around planets, comets, binary stars

① The planets move in elliptical orbits with the Sun at one focus

Ellipticity e : $b^2 = a^2(1-e^2)$ 	a semi-major axis
b " -minor

② A line from the Sun to a planet sweeps out equal areas in equal times

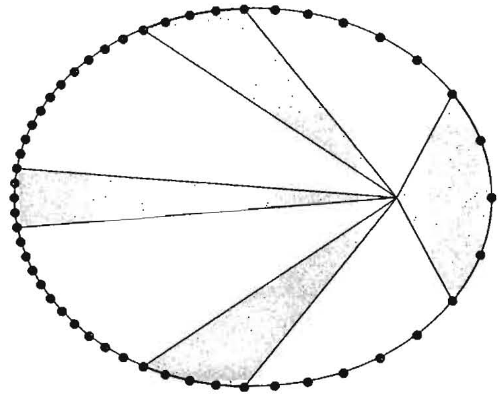


FIGURE 2.2 Kepler's second law states that the area swept out by a line between a planet and the focus of an ellipse is always the same for a given time interval, regardless of the planet's position in its orbit. The dots are evenly spaced in time.

Kepler's first and second laws are illustrated in Fig. 2.2, where each dot on the ellipse represents the position of the planet during evenly spaced time intervals.

Kepler's third law was published ten years later in the book *Harmonica Mundi* (*The Harmony of the World*). His final law relates the average orbital distance of a planet from the Sun to its sidereal period:

Kepler's Third Law *The Harmonic Law.*

$$\boxed{P^2 = a^3}$$

where P is the orbital period of the planet, measured in *years*, and a is the average distance of the planet from the Sun, in *astronomical units*, or AU. An **astronomical unit** is, by definition, the average distance between Earth and the Sun, 1.496×10^{11} m. The graph of Kepler's third law shown in Fig. 2.3 was prepared using data for each planet in our Solar System as given in Appendix C.

In retrospect it is easy to understand why the assumption of uniform and circular motion first proposed nearly 2000 years earlier was not determined to be wrong much sooner; in most cases, planetary motion differs little from purely circular motion. In fact, it was actually fortuitous that Kepler chose to focus on Mars, since the data for that planet were particularly good and Mars deviates from circular motion more than most of the others.

The Geometry of Elliptical Motion

To appreciate the significance of Kepler's laws, we must first understand the nature of the **ellipse**. An ellipse (see Fig. 2.4) is defined by that set of points that satisfies the equation

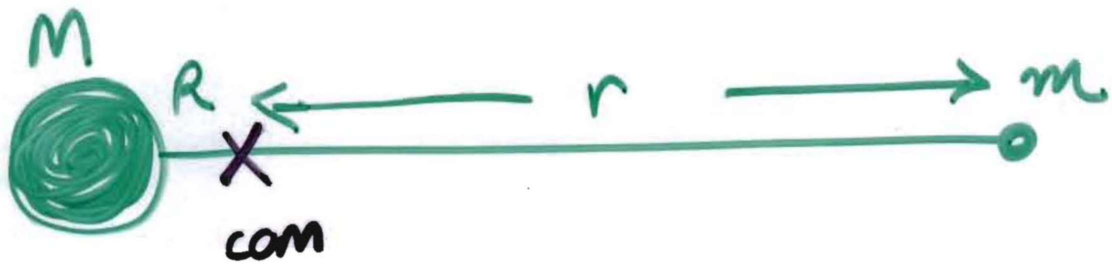
$$r + r' = 2a, \tag{2.1}$$

where a is a constant known as the **semimajor axis** (half the length of the long, or major axis of the ellipse), and r and r' represent the distances to the ellipse from the two **focal**

Center of mass

The Earth and the Sun orbit around their center of mass.

Putting the center of mass at the origin:



$$MR = -mr \; ; \; \mu = \frac{mM}{m+M}$$

So a more correct statement of Kepler's first law is:

Each planet moves on an elliptical orbit with the center of mass at one focus.

Q The Sun has mass 2×10^{30} kg

Earth " " 6×10^{24} kg

1 AU = 1.5×10^{11} m

The Sun's radius is 7×10^{8} m

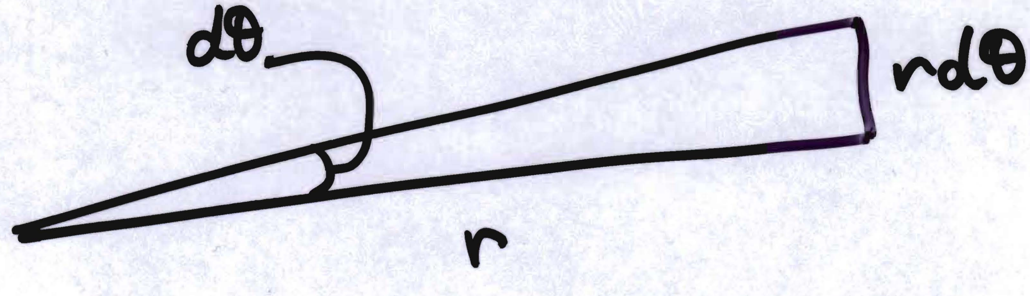
Is the Sun-Earth center of mass inside the Sun or outside?

$\boxed{Q}$ Prove Kepler's second law, using conservation of angular momentum

Hint : what area is swept out in time dt ?

Hint : Angular momentum $L = \vec{r} \times \vec{p}$

In time dt, planet moves an angle $d\theta$ along its orbit.



area of wedge is $\frac{1}{2} r \cdot r\,d\theta$

$\cancel{\frac{1}{2} r}$

Distance moved $= v_\theta \, dt = r\,d\theta$

So area $= \frac{1}{2} r \cdot v_\theta \, dt$

Angular momentum $\vec{L} = m \, \vec{r} \times \vec{v}$

$L = m r v_\theta$

Conservation of angular momentum

$\Rightarrow$ at any point $m_1 r_1 v_{\theta_1} = m_2 r_2 v_{\theta 2}$

Planet's mass unchanged, so

$$r_1 v_{\theta 1} = r_2 v_{\theta 2}$$

In time t:

$$\text{area} = \int_0^{t} \frac{1}{2} r \, v_\theta \, dt$$

Since $r v_\theta$ doesnt change along the orbit, neither does the area.

ENERGY OF ORBITS

Q

What are the two major contributions to the Earth's orbital energy?

Kinetic energy $\boxed{K = \frac{1}{2} m v^2}$

Gravitational potential energy

Since the gravitational force is a central force

Energy is conserved & we can define a potential energy.

What is the work involved in pushing a planet away from the Sun?

Vector notation

$$\Delta U = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

Using gravitational force law

&

fact that $\vec{F}$ and $\vec{r}$ are in same direction

$$\Delta U = \int_{r_i}^{r_f} \frac{GMm}{r^2}\, dr$$

Integrating, we find that

$$U_f - U_i = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

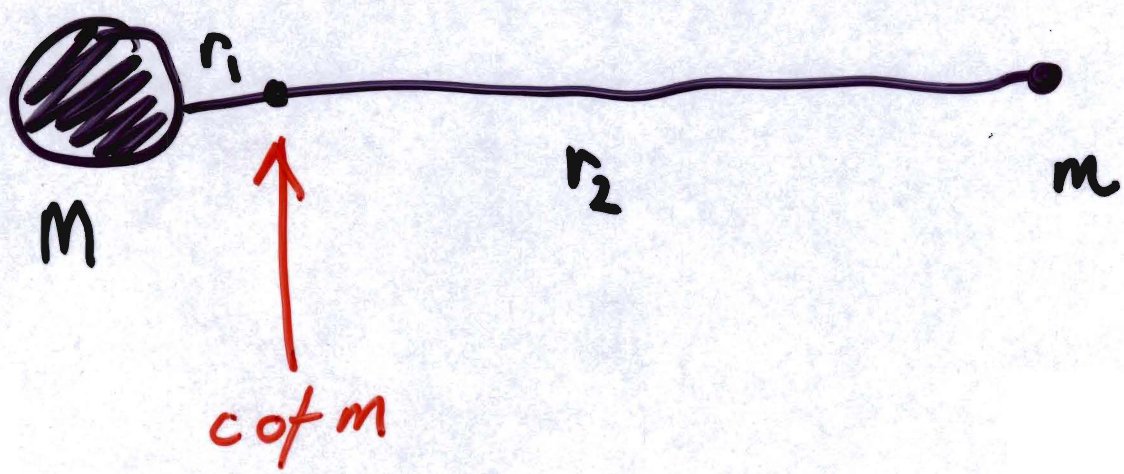
Pushing all the way to ∞ and <u>defining</u> $U(\infty) = 0$

$$\boxed{U = -\frac{GMm}{r}}$$

Now we use the conservation of energy ($K + U = \text{const}$)

So total energy $= K + U$

Looking at an asteroid in orbit around the Sun



Total energy of the system

$$E_{tot} = \frac{1}{2} M v_{Sun}^2 + \frac{1}{2} m v_{asteroid}^2 - \frac{GMm}{(r_1+r_2)}$$

Define $r_1 + r_2 = r$

$$v_{asteroid} = v$$

$\boxed{Q}$ We can ignore the term for the Sun's k.e. here. Why?

$$\text{velocity} = \frac{2\pi r}{P} \qquad P \text{ is period (same for both)}$$

$$\frac{\text{k.e. (Sun)}}{\text{k.e. (asteroid)}} = \frac{M v_\odot^2}{m v^2}$$

$$\frac{v_\odot}{v} = \frac{r_1}{r_2} \quad \text{and} \quad \frac{r_1}{r_2} = \frac{m}{M}$$

(center of mass)

$$\text{So } \frac{\text{k.e. } \odot}{\text{k.e. (asteroid)}} = \frac{M}{m} \cdot \frac{m^2}{M^2}$$

$$= \frac{m}{M} \ll 1$$

$$\text{So } E_{tot} = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

For a circular orbit

$$v = \frac{2\pi r}{P}$$

Kepler's 3rd law in full form :

$$\boxed{P^2 = \frac{4\pi^2 r^3}{GM}}$$

($a \leftrightarrow r$ here)

$$\text{So } v^2 = \frac{4\pi^2 r^2}{P^2} = \frac{GM}{r}$$

$$\text{and kinetic energy} = \frac{1}{2} m v^2 = \frac{GmM}{2r}$$

(same formula applies for ellipse $\bar{c}$ semi-major axis $a \leftrightarrow r$)

We can now simplify formula for total energy

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$= \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\boxed{E_{tot} = -\frac{GMm}{2r}}$$

E_{tot} -ve $\Rightarrow$ bound

Formula holds for any bound orbit, using a (semi-major axis) for r.

$\rightarrow$ E_{tot} depends only on a, not eccentricity

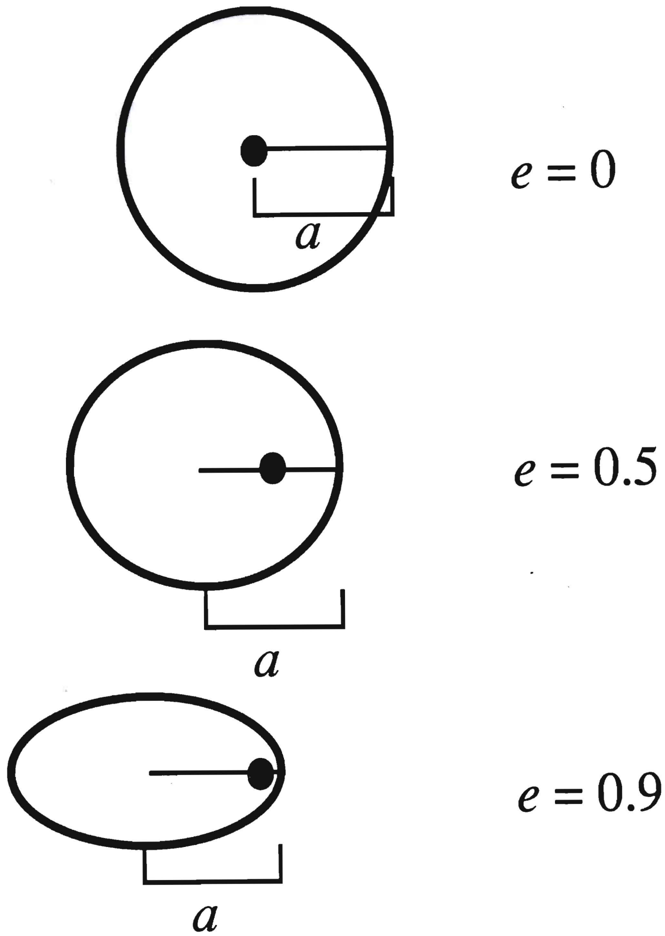


FIGURE 9.7 Three orbits, with the same semimajor axis but different eccentricities, have the same amount of orbital energy.

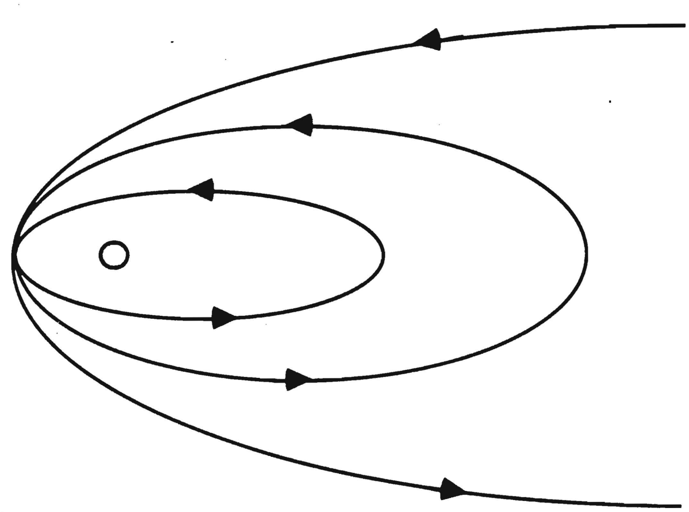


FIGURE 9.8 Closed orbits are in the shape of ellipses; as the energy increases, the orbit stretches out towards infinity until the orbit is a parabola and the body escapes.

For all bound orbits

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

present radius

semi-major axis of ellipse

→ mass m cancels : orbit is the same for Jupiter or a matchbox, depends on M (Sun's mass) and a

→ <u>total</u> orbital energy doesn't change but K & U trade off.

General formula for circular velocity

$$v = \sqrt{\frac{GM}{r}}$$

We can use these formulae to work out speed & period of Hubble Space telescope, in its low-earth orbit (600 km from surface)

$$v_c = \sqrt{\frac{G\,m_\oplus}{r}}$$

$$= \sqrt{\left(\frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{(6378 + 600) \times 10^3}\right)}$$

$$= 7.6 \text{ km/s}$$

$$P = \frac{2\pi r}{v_c} = 96 \text{ min.}$$

Escape velocity

Take a satellite in orbit about the Earth.

If it burns fuel & increases total energy, eventually total energy will be zero and it will no longer be bound to Earth

$$\text{k.e} + \text{p.e} = 0$$

$$\frac{1}{2} m v^2 - \frac{G M m}{r} = 0$$

$$v^2 = \frac{2GM}{r}$$

or $$v_e = \sqrt{\frac{2GM}{r}}$$

At surface of Earth $v_e = 11$ km/s

Escape velocity from Solar System @ 1 AU is

$$\sqrt{\frac{2 G M_\odot}{1 \text{ AU}}}$$

4.2×10^6 cm/s

$= 4.2 \; 10^4$ m/s km/s

42 km/s

Escape velocity from Solar System at surface of Sun is:

For comparison, the Sun's orbital motion about the center of the Galaxy is ~ 220 km/s.

Synchronous satellites

Escape velocity is $\sqrt{2} \times$ circular velocity @ same r.

$\odot^r$ velocity $\quad v_c = \sqrt{\frac{GM}{r}}$

Synchronous satellite: takes 1 day to complete orbit, going in same direction as Earth's rotation, so appears fixed above one point

Distance from Earth ?

Kepler's 3rd law $\quad p^2 = \dfrac{4\pi^2 a^3}{G M_{earth}}$.

Period = 1 day = 24 x 3600s

solve for a : $\quad a^3 = \dfrac{p^2 \times G \times M_{earth}}{4\pi^2}$

$$= \frac{(24 \times 3600)^2 \times 6.7 \times 10^{-8} \times 6 \times 10^{27}}{4\pi^2}$$

$a = 4.2 \times 10^9$ cm or 42,000 km

Earth's radius is 6378 km so this is about 35,600 km above surface

Space shuttle orbits about 300 km above surface – what is its period ?

Kepler's second law revisited

Center of mass frame : now use m_1, m_2

$\vec{r}_1, \vec{r}_2$

$\vec{v}_1, \vec{v}_2$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Total orbital angular momentum of system :

$$\vec{L} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

Reduces to $\vec{L} = \mu \vec{r} \times \vec{v}$

Reduced mass moving about Total mass M located @ origin

Then Kepler's 2nd becomes $\dfrac{dA}{dt} = \dfrac{1}{2} \dfrac{L}{\mu}$

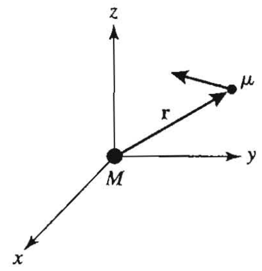


FIGURE 2.12 A binary orbit may be reduced to the equivalent problem of calculating the motion of the reduced mass, μ, about the total mass, M, located at the origin.

becomes

$$\mathbf{L} = \mu \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{p}, \tag{2.26}$$

where $\mathbf{p} \equiv \mu \mathbf{v}$. The total orbital angular momentum equals the angular momentum of the reduced mass only. *In general, the two-body problem may be treated as an equivalent one-body problem with the reduced mass μ moving about a fixed mass M at a distance r* (see Fig. 2.12).

The Derivation of Kepler's First Law

To obtain Kepler's laws, we begin by considering the effect of gravitation on the orbital angular momentum of a planet. Using center-of-mass coordinates and evaluating the time derivative of the orbital angular momentum of the reduced mass (Eq. 2.26) give

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F},$$

the second expression arising from the definition of velocity and Newton's second law. Notice that because $\mathbf{v}$ and $\mathbf{p}$ are in the same direction, their cross product is identically zero. Similarly, since $\mathbf{F}$ is a central force directed inward along $\mathbf{r}$, the cross product of $\mathbf{r}$ and $\mathbf{F}$ is also zero. The result is an important general statement concerning angular momentum:

$$\frac{d\mathbf{L}}{dt} = 0, \tag{2.27}$$

the angular momentum of a system is a constant for a central force law. Equation (2.26) further shows that the position vector $\mathbf{r}$ is always perpendicular to the constant angular momentum vector $\mathbf{L}$, meaning that the orbit of the reduced mass lies in a plane perpendicular to $\mathbf{L}$.

Using the radial unit vector $\hat{\mathbf{r}}$ (so $\mathbf{r} = r\hat{\mathbf{r}}$), we can write the angular momentum vector in an alternative form as

$$\begin{aligned} \mathbf{L} &= \mu \mathbf{r} \times \mathbf{v} \\ &= \mu r \hat{\mathbf{r}} \times \frac{d}{dt}(r\hat{\mathbf{r}}) \end{aligned}$$

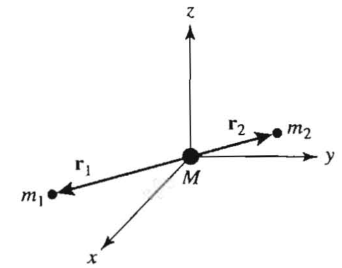


FIGURE 2.11 The center-of-mass reference frame for a binary orbit, with the center of mass fixed at the origin of the coordinate system.

Next, define the **reduced mass** to be

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1 + m_2}.} \tag{2.22}$$

Then $\mathbf{r}_1$ and $\mathbf{r}_2$ become

$$\mathbf{r}_1 = -\frac{\mu}{m_1}\mathbf{r} \tag{2.23}$$

$$\mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r}. \tag{2.24}$$

The convenience of the center-of-mass reference frame becomes evident when the total energy and orbital angular momentum of the system are considered. Including the necessary kinetic energy and gravitational potential energy terms, the total energy may be expressed as

$$E = \frac{1}{2}m_1 |\mathbf{v}_1|^2 + \frac{1}{2}m_2 |\mathbf{v}_2|^2 - G\frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}.$$

Substituting the relations for $\mathbf{r}_1$ and $\mathbf{r}_2$, along with the expression for the total mass of the system and the definition for the reduced mass, gives

$$E = \frac{1}{2}\mu v^2 - G\frac{M\mu}{r}, \tag{2.25}$$

where $v = |\mathbf{v}|$ and $\mathbf{v} \equiv d\mathbf{r}/dt$. We have also used the notation $r = |\mathbf{r}_2 - \mathbf{r}_1|$. The total energy of the system is equal to the kinetic energy of the reduced mass, plus the potential energy of the reduced mass moving about a mass M, assumed to be located and fixed at the origin. The distance between μ and M is equal to the separation between the objects of masses m_1 and m_2.

Similarly, the total orbital angular momentum,

$$\mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2$$