Orbits and Kepler’s Laws

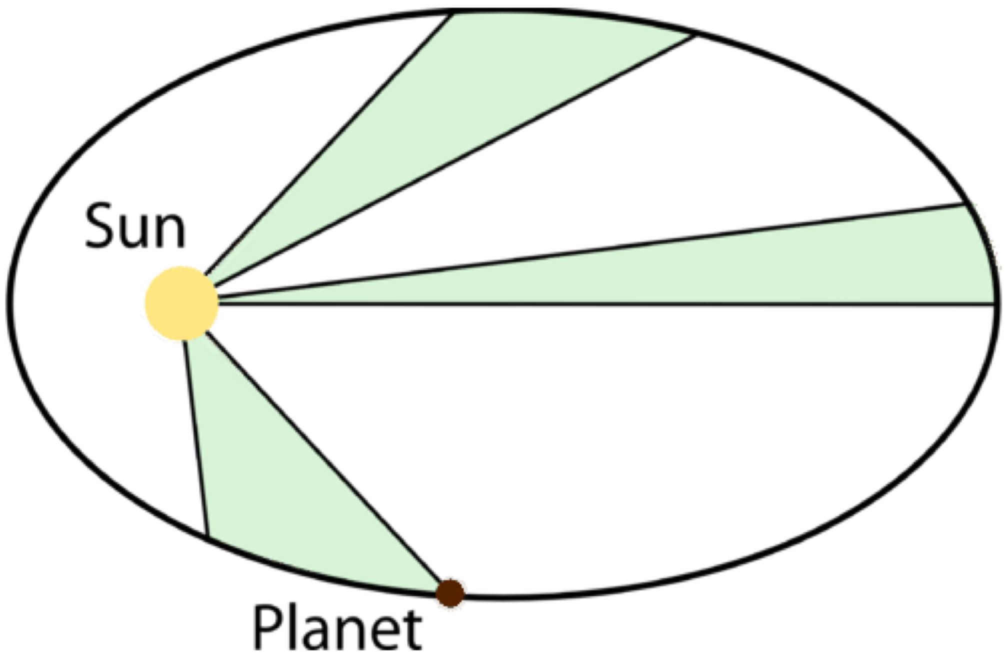
- Kepler's laws: apply to planets around the Sun, moons around planets, comets, binary stars

- The planets move in elliptical orbits with the Sun at one focus:

  - $Ellipticity\ (e)$: $b^2 = a^2(1 - e^2)$
  - a is the semi-major axis
  - b is the semi-minor axis

Orbits and Kepler’s Laws

2. A line from the Sun to a planet sweeps out equal areas in equal times.



Orbits and Kepler's Laws

3. More distant planets orbit the Sun at slower average speeds, obeying a precise mathematical relationship.

- $The\ Harmonic\ Law: p^2 = a^3$

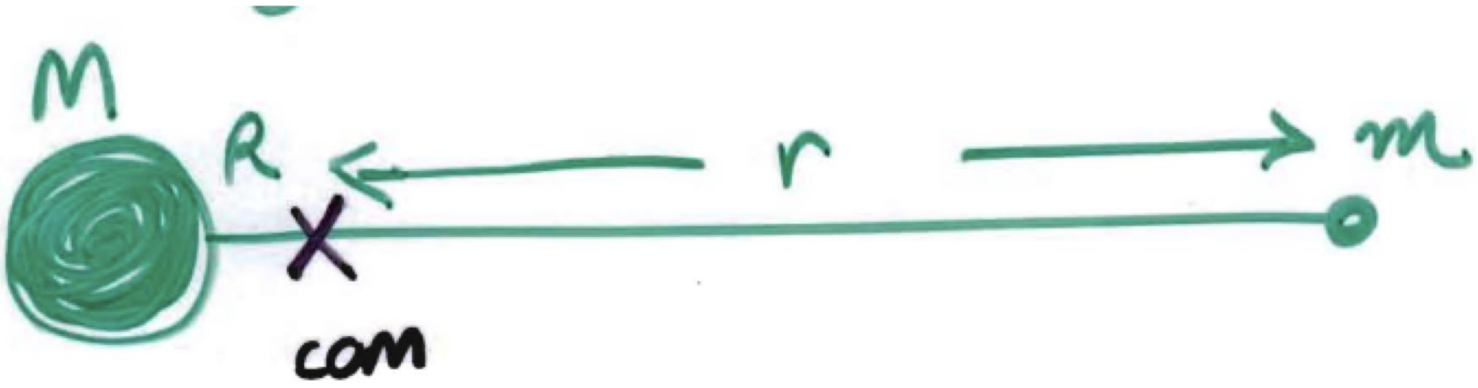
- *p is the planet's orbital period in years*

- *a is its average distance from the sun in Astronomical Units (AU)*

Center of Mass

The Earth and the Sun both orbit around their common center of mass.

Putting the center of mass at the origin:



$$MR = -mr;$$

Reduced mass $\mu = \frac{mM}{m+M}$

Center of Mass

So a more correct statement of Kepler's first law is:

Each planet moves on an elliptical orbit with the center of mass at one focus.

Question:

The Sun has a mass of 2×10^{30} kg and the Earth has a mass of 6×10^{24} kg.

1 AU = 1.5×10^{11} m; The Sun's radius is 7×10^{8} m

Is the Sun-Earth's center of mass inside the Sun or outside?

Question:

Prove Kepler's second law, using conservation of angular momentum.

(Use a circular orbit for simplicity)

- Hint: draw a diagram

- Hint: What area is swept out in time dt?

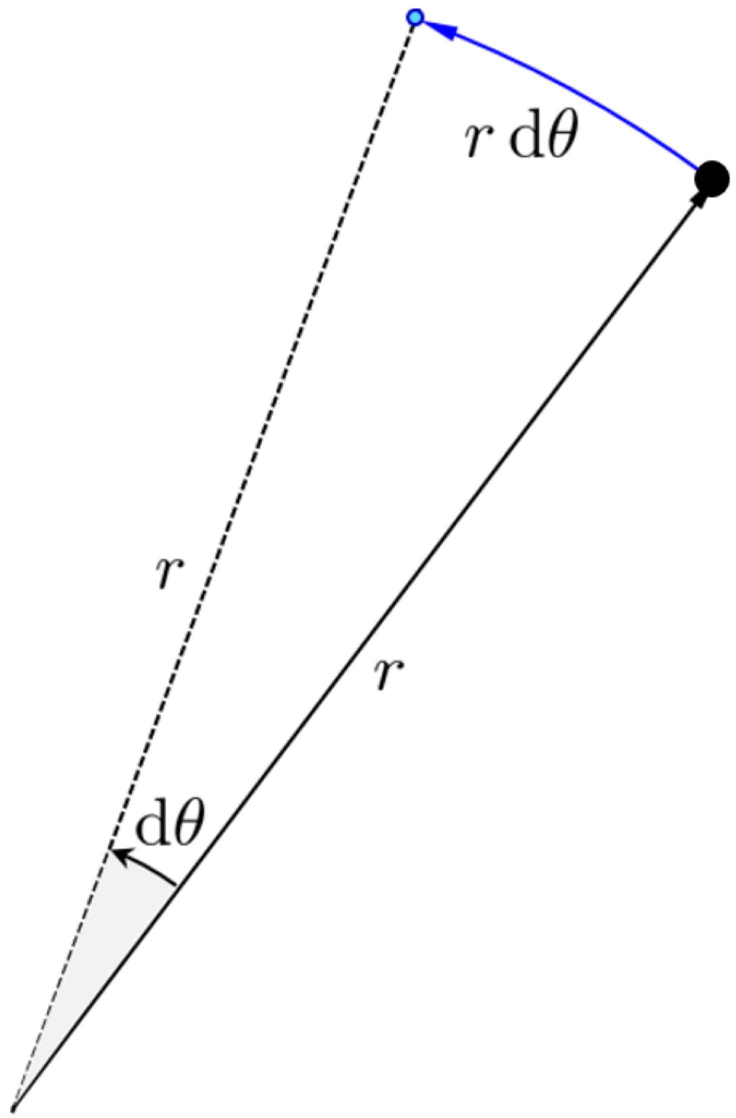
- Hint: Angular momentum $L = \vec{r} \times \vec{p}$

In time dt, a planet moves an angle dϴ along its orbit.

Area of wedge is $\frac{1}{2}r \cdot rd\theta$

Distance moved = $v_\theta dt = rd\theta$

So area $= \frac{1}{2}r \cdot v_\theta dt$



Angular momentum: $\vec{L} = m\vec{r}\times\vec{v}$

$$L = mrv_\theta$$

Conservation of angular momentum at any point:

$$m_1 r_1 v_{\theta_1} = m_2 r_2 v_{\theta_2}$$

The planet's mass is unchanged, so:

$$r_1 v_{\theta_1} = r_2 v_2$$

In time t:

$$area = \int_0^t \frac{1}{2} r v_\theta \, dt$$

Since $r v_\theta$ doesn't change along the orbit, neither does the area.

So Kepler's second law is simply a statement that angular momentum is conserved in the Solar System

Energy of Orbits

Question:

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Answer:

Kinetic and potential energy

Kinetic energy:

$$K = \frac{1}{2} m v^2$$

Gravitational Potential Energy:

- Since the gravitational force is a central force, energy is conserved and we can define potential energy.

Derive formula for gravitational p.e:

- What is the work involved in pushing a planet away from the Sun?

Using vector notation:

$$\Delta U = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

Using the gravitational force law and the fact that $\vec{F}$ and $\vec{r}$ are in the same direction:

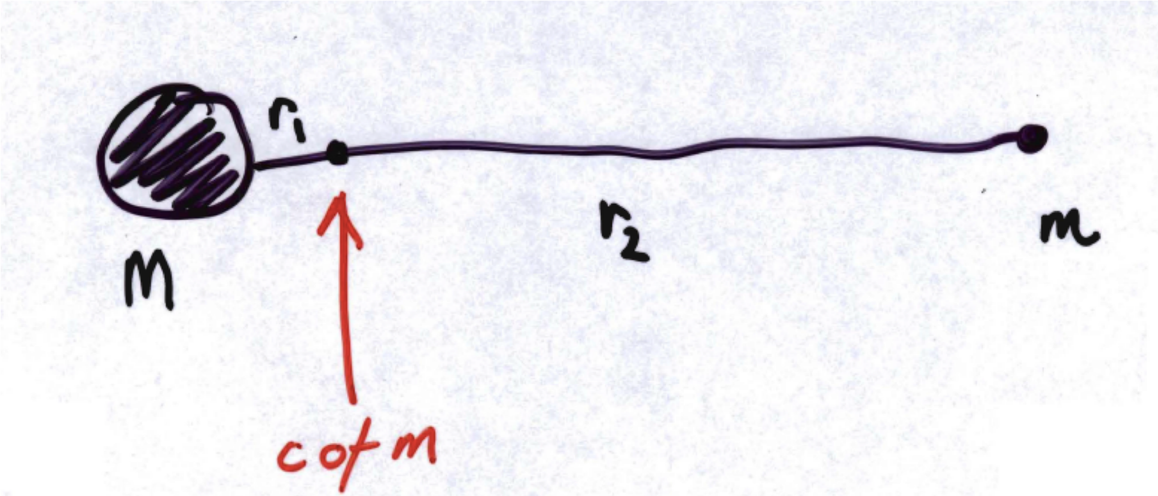
$$\Delta U = \int_{r_i}^{r_f} \frac{GMm}{r^2} dr$$

Integrating: $U_f - U_i = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$

Pushing all the way to ∞ and <u>defining</u> $U(\infty) = 0$

$$\boxed{U = -\frac{GMm}{r}}$$

Looking at an asteroid in orbit around the Sun:



Total energy of the system:

$$E_{tot.} = \frac{1}{2} M v_{Sun}^2 + \frac{1}{2} M v_{asteroid}^2 - \frac{GMm}{(r_1 + r_2)}$$

Define $r_1 + r_2 = r$ $\qquad v_{asteroid} = v$

Question:

We can ignore the term for the Sun's K.E. here. Why?

Answer:

$$Velocity = \frac{2\pi r}{p}$$

P is the period (same for both)

So $\frac{K.E.(Sun)}{K.E.(\,asteroid)} = \frac{Mv_{\odot}^2}{mv^2}$

$$\frac{v_{\odot}}{v} = \frac{r_1}{r_2} \ \& \ \frac{r_1}{r_2} = \frac{m}{M} \ (\,center\ of\ mass)$$

Therefore $\frac{K.E._{\odot}}{K.E.(\,asteroid)} = \frac{m}{M} \cdot \frac{m^2}{M^2} = \frac{m}{M} \ll 1$

So $E_{Tot.} = \frac{1}{2}mv^2 - \frac{GMm}{r}$

For a circular orbit:

$$v = \frac{2\pi r}{p}$$

Kepler's third law in full form:

$$p^2 = \frac{4\pi^2 r^3}{GM}, (a \leftrightarrow r\ here)$$

We can use this formula to estimate the mass of an object once we have something orbiting it

So $v^2 = \frac{4\pi^2 r^2}{p^2} = \frac{GM}{r}$

And $kinetic\ energy = \frac{1}{2}mv^2 = \frac{GmM}{2r}$

(Same formula applies for ellipse, semi-major axis $a \leftrightarrow r$)

We can now simplify the formula for the total energy:

$$E_{Tot.} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

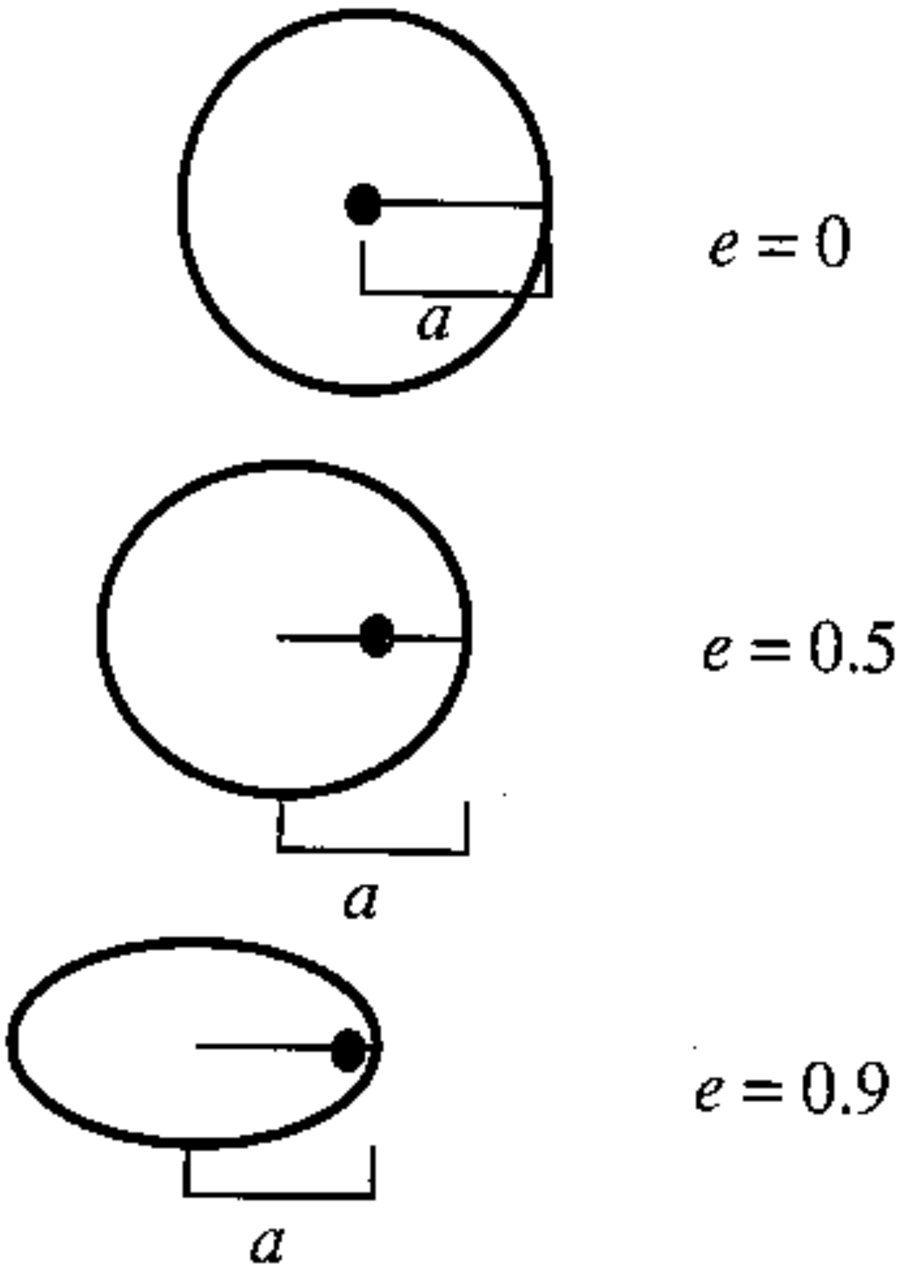
$$E_{Tot.} = \frac{gMm}{2r} - \frac{GMm}{r}$$

$$E_{Tot.} = \frac{-GMm}{2r}$$

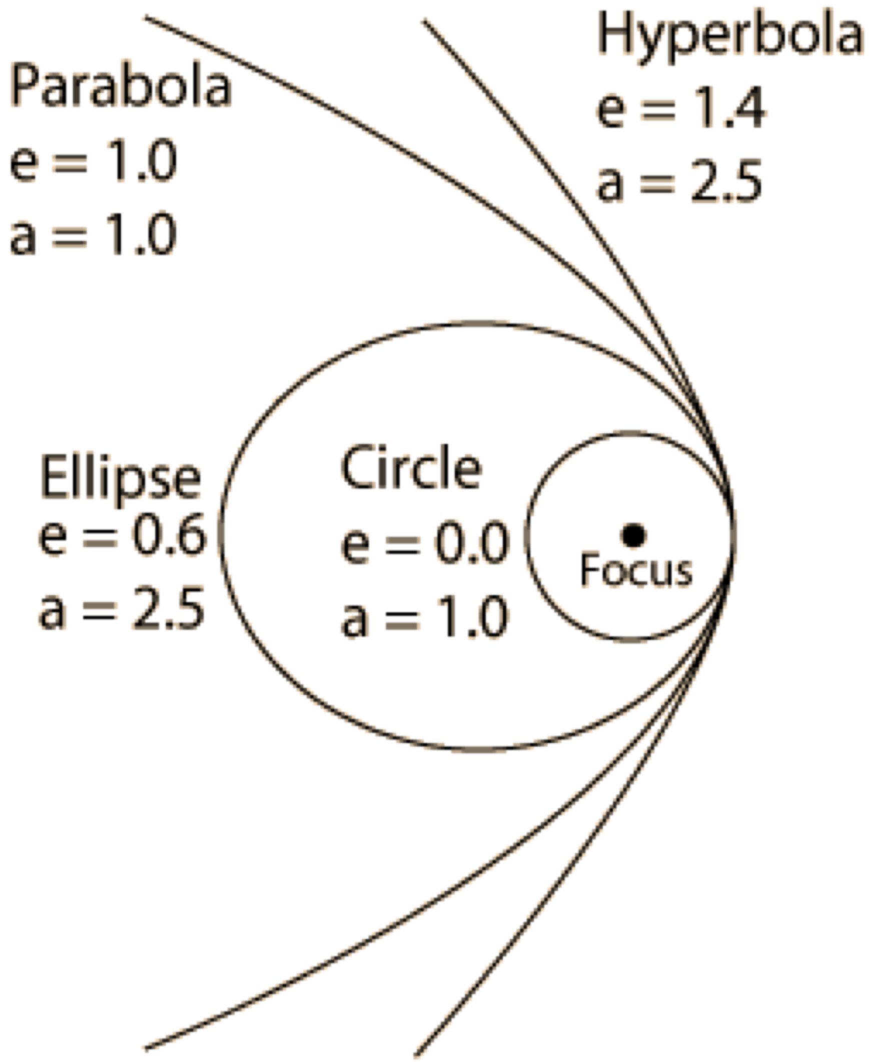
$$E_{Tot.} < 0 \rightarrow bound$$

Formula holds for any bound orbit, using a (semi-major axis) for r.

$\rightarrow E_{Tot.}$ Depends only on a, not eccentricity.



Three orbits, with the same semi-major axis but different eccentricities, have the same amount of orbital energy.



Closed orbits are in the shape of ellipses; as the energy increases, the orbit stretches out towards infinity until the orbit is a parabola and the body escapes.

For all bound orbits:

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

Where r is the present radius,

And a is the semi-major axis of the ellipse.

Mass m cancels: orbit is the same for Jupiter or a matchbox, it depends on M (Sun's mass) and a.

Total orbital energy doesn't change but K and U trade off.

General formula for circular velocity:

$$v = \sqrt{\frac{GM}{r}}$$

You can use these formulae to work out speed and period of the Hubble Space Telescope, in its low-Earth orbit (600 km from the surface).

$$v_{circ.} = \sqrt{\frac{GM_{Earth}}{r}}$$

$$= \sqrt{\frac{6.67\times10^{-11}\times6\times10^{24}}{(6378 + 600)\times10^{3}}}$$

=7.6 km/s

$$p = \frac{2\pi r}{v_c} = 96\,min.$$

Escape Velocity:

Take a satellite in orbit about the earth. If it burns fuel and increases total energy, eventually total energy will be zero and it will no longer be bound to the earth.

$$K.E. + P.E. = 0$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

$$v^2 = \frac{2GM}{r}$$

$$or\ v_{esc.} = \sqrt{\frac{2GM}{r}}$$

At the surface of Earth: $v_e = 11\ km/s$

Escape velocity from the solar system at 1AU:

$$\sqrt{\frac{2GM_{\odot}}{1AU}} = 4.2\times10^{6}\,\frac{cm}{s} = 4.2\times10^{4}\,\frac{m}{s} = 42\,\frac{km}{s}$$

Question:

What is the escape velocity from the solar system at the surface of the Sun ?

For comparison, the Sun's orbital motion about the center of the Galaxy is~ 220 km/s.

Synchronous Satellites:

Escape velocity is $\sqrt{2}\times$ circular velocity at the same r.

$$\odot^r \ \ velocity:\ v_c = \sqrt{\frac{GM}{r}}$$

Synchronous satellite:

Takes 1 day to complete ITS orbit, going in same direction as Earth's rotation, so it appears fixed above one point.

Distance from Earth?

Kepler's third law: $p^2 = \frac{4\pi^2 a^3}{GM_{earth}}$

Period = 1 day = 24×3600s

Solve for a: $a^3 = \frac{p^2 \times g \times M_{earth}}{4\pi^2} = \frac{(24\times3600)^2 \times 6.7\times10^{-8} \times 6\times10^{27}}{4\pi^2}$

$$a = 4.2\times10^9\ cm\ or\ 42{,}000\ km$$

Earth's radius is 6378 km, so this is about 35,600 km above the surface.

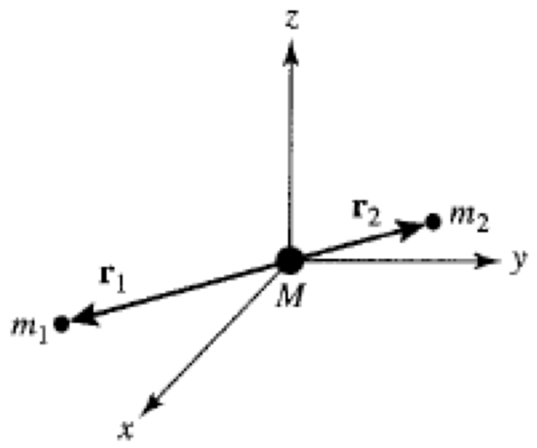
Quite different from low-Earth orbit!

Kepler's second law revisited:

Working in the center of mass frame

Now use $m_1, m_2, \vec{r}_{1,} \vec{r}_2, \vec{v}_1, \vec{v}_2$

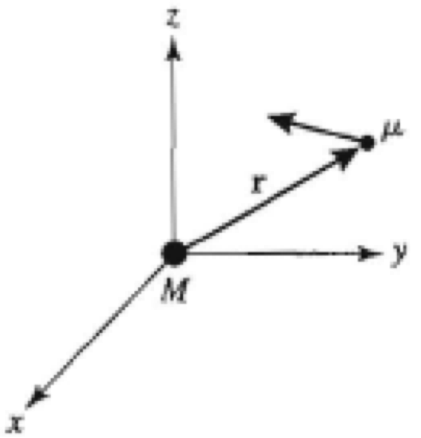
Reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$



$$\boxed{\begin{aligned} \mathbf{r}_1 &= -\frac{\mu}{m_1}\mathbf{r} \\ \mathbf{r}_2 &= \frac{\mu}{m_2}\mathbf{r}. \end{aligned}}$$

Total orbital angular momentum of the system:

$$\vec{L} = m_1\vec{r}_1\times\vec{v}_1 + m_2\vec{r}_2\times\vec{v}_2$$



This reduces to:

$$\vec{L} = \mu\vec{r}\times\vec{v}$$

In general, the two-body problem may be treated as a one-body problem with the reduced mass mu moving about total mass M located at radius r