

Black holes

Objects so incredibly dense that

- neither electron nor neutron degeneracy pressure can hold them up against the pull of gravity
- they twist the nature of space-time
 - photons, matter - nothing can get away
ie escape velocity is greater than the speed of light

Review of Special Relativity

(see Kutner Ch 7)

Inertial reference frames, relative velocity v .

- Time dilation $t = t_0 / \sqrt{1 - v^2/c^2}$

$$t_0 \text{ 'proper time'} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- Length contraction $\ell = \ell_0 \sqrt{1 - v^2/c^2}$

- Relativistic Doppler shift

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} \quad \lambda' = \lambda + \Delta\lambda$$

- Relativistic momentum $p = \gamma m \tilde{v}$

- Total energy $E = \gamma mc^2$

$$(\text{rest energy} = mc^2)$$

Curved spacetime

Q

How do you measure distance
in a very uneven, hilly terrain ?

How might you extend this to
curved spacetime ?

Measure length by adding together
small elements ds (in space) or
 ds (spacetime)

2-D Cartesian coords

$$dl^2 = dx^2 + dy^2 \quad \text{"metric"}$$

$$\text{length} = \int_1^2 \sqrt{dl^2}$$

$$= \int_1^2 \sqrt{(dx^2 + dy^2)} \text{ along path}$$

In a curved space, we can use the same technique of integrating along the path to get the distance

In spacetime we have a time coordinate too

metric for flat spacetime, or interval ds between events A,B

$$ds^2 = (\text{distance travelled by light})^2 - (\text{distance between events A\&B})^2$$
$$= c^2 dt^2 - dx^2 - dy^2 - dz^2$$



Q Under what circumstances
~~can~~ will the interval ds^2 between two events be

- (a) zero ?
- (b) ~~positive~~ ?
- (c) negative ?

- Space interval = 0
(eg you sit in your chair for 10 mins)

$$dx = dy = dz = 0$$

$$\text{So } ds^2 = c^2 dt^2 > 0$$

Interval called TIMELIKE

- Photon travels from A to B

$$\frac{\text{distance elapsed}}{\text{time elapsed}} = c$$

$$\text{so } c^2 dt^2 = dx^2 + dy^2 + dz^2$$

$$\text{So } ds^2 = 0$$

Interval called LIGHTLIKE

- No time elapsed

$$dt = 0$$

$$\begin{aligned} \text{So } ds^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= -dx^2 - dy^2 - dz^2 \end{aligned}$$

$$< 0$$

Such an interval is called ~~LIGHTLIKE~~ **SPACELIKE**

Spherical coordinates in flat space

$$\begin{aligned} ds^2 &= c^2 dt^2 - dr^2 - (r d\theta)^2 - \\ &\quad (r \sin \theta d\phi)^2 \end{aligned}$$

The metric has the same value
whatever coordinate system you use
allows us to express physical laws

which take the same form
whatever coordinate system (reference
frame) you choose

The metric tells us how spacetime
curves

Can derive all of special relativity
from this metric

Special relativity allows us to
express physical laws in ways that
are the same for all inertial frames

(moving at constant speed with
respect to one another)

Lorentz Transformations

Give the rules by which you can transform between one frame moving at a speed v with respect to another.

Can derive the Lorentz transformation by assuming that the interval $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ is invariant (the same when measured in any inertial frame)

Example

Imagine a rocket moving at speed v thru a (very long) lab, in the x direction. x, t lab coords
 x', t' rocket coords

Can derive other Lorentz transformations in similar way.

Good book on special relativity :

"Spacetime Physics" by Taylor & Wheeler

Time dilation & simultaneity

Two events may be simultaneous in one frame of reference & not the other.

interval

'Proper time' τ is time measured by a clock at rest between 2 events

$$\Delta t_{\text{moving}} = \frac{\Delta t_{\text{rest}}}{\sqrt{1 - v^2/c^2}}$$

There is no absolute time, same \forall observers

Q

Cosmic rays in space collide with particles in the upper atmosphere and produce muons. Muons are unstable, decay with average lifetime $\approx 2.2 \times 10^{-6}$ s. (proper time).

They are moving at $v = 0.995c$

How long would an observer on the ground see muons survive for?

A

$$\sqrt{1 - \frac{v^2}{c^2}} = 0.0999 \text{ so}$$

$$dt_{\text{moving}} = \frac{2 \times 10^{-6} \text{ s}}{0.0999} = 2 \times 10^{-5} \text{ s}$$

This has been verified experimentally to high accuracy

General Relativity

Describes how intervals in spacetime
are measured in the presence of mass

"gravitational field curves spacetime"

Curved spacetime determines trajectory
of particles (including photons)

Also, time runs more slowly in
curved spacetime

Tests of general relativity

- (i) Elliptical orbits close to a large mass
pass through regions of different

spatial curvature — general relativity predicts that such orbits will not be closed (will precess) For Mercury, Einstein calculated the amount of precession due to this ($43''/\text{century}$) and it was exactly right.

(ii) Bending of light by gravitational field
In a curved space, a geodesic is the shortest distance between two points



Q What are geodesics on the two-dimensional curved spaces

(a) on the surface of a sphere

[(b) on a saddle ?]

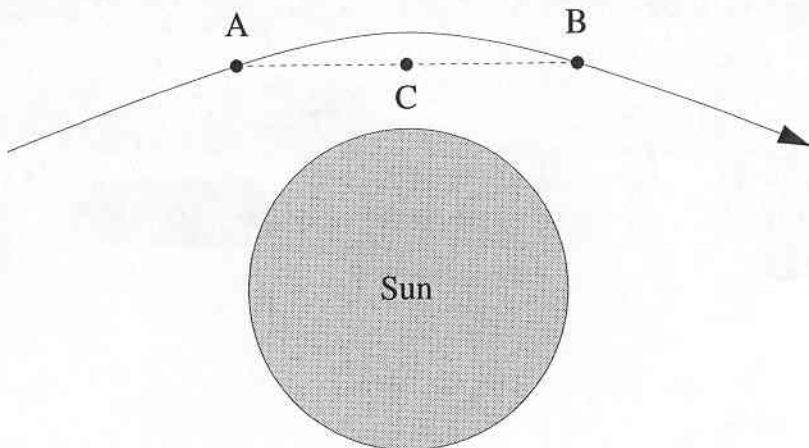


Figure 16.3 A photon's path around the Sun is shown by the solid line. The bend in the photon's trajectory is greatly exaggerated.

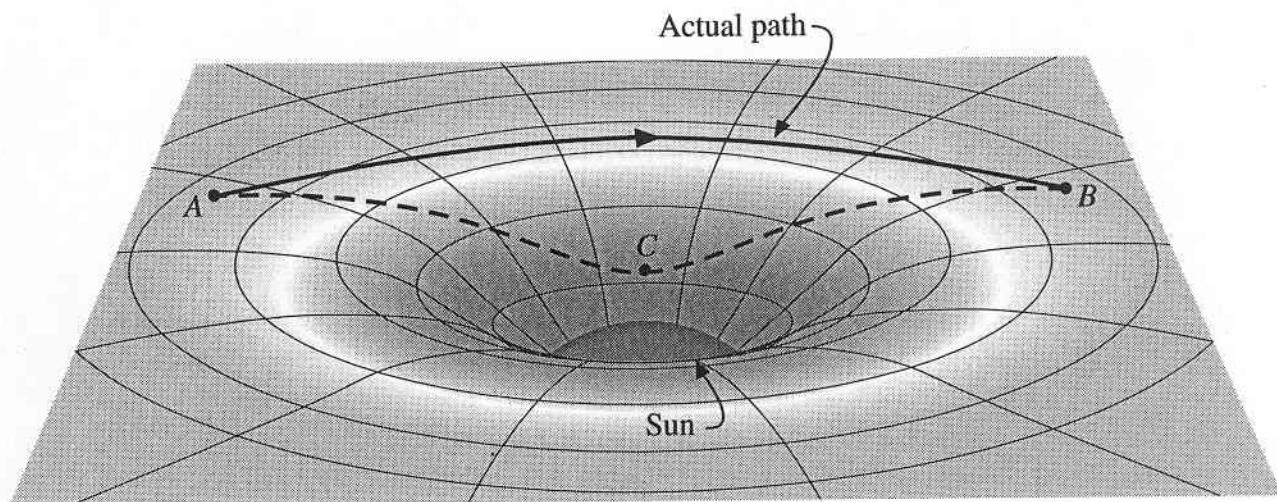


Figure 16.4 Comparison of two photon paths through curved space between points A and B.

Nothing can move faster than
the speed of light

light travels on geodesics in
curved space

thus light from a star that appears
close to the Sun will bend

Einstein predicted the amount of
this bending it was confirmed
on the eclipse expedition of 1919

This was impressive confirmation of
theory as data might have falsified
it ("risky prediction")

(iii) gravitational time dilation

In a strong gravitational field, time runs more slowly

In 1976 Mars was close to conjunction with the Sun. Radio signals from Viking spacecraft on Mars were delayed as they travelled through the curved space close to the Sun

PRINCIPLE OF EQUIVALENCE

Gravitational and inertial mass are the same

→ tested to one part in 10^{-12}

Around-the-World Atomic Clocks

In October 1971, Hafele and Keating flew cesium beam atomic clocks around the world twice on regularly scheduled commercial airline flights, once to the East and once to the West. In this experiment, both gravitational time dilation and kinematic time dilation are significant - and are in fact of comparable magnitude. Their predicted and measured time dilation effects were as follows:

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Predicted:	Time difference in ns	
	Eastward	Westward
Gravitational	144 +/- 14	179 +/- 18
Kinematic	-184 +/- 18	96 +/- 10
Net effect	-40 +/- 23	275 +/- 21
Observed:	-59 +/- 10	273 +/- 21

Gravitational calculation

Kinematic calculation

Recall metric for a flat space in spherical coordinates :

$$ds^2 = c^2 dt^2 - dr^2 - (r d\theta)^2 - (r \sin \theta d\phi)^2$$

Schwarzschild solved Einstein's general relativistic field equation (how matter curves space, how things move in curved space) for simple case of curved spacetime surrounding a spherical mass M (as a black hole)

$$ds^2 = \left(c dt \sqrt{1 - \frac{2GM}{rc^2}} \right)^2 - \left(\frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \right)^2 - (r d\theta)^2 - (r \sin \theta d\phi)^2$$

We set this up with ~~coordinates~~^{coordinates}
 r, θ, ϕ & t used by an observer
at rest a large distance from
the origin where the mass M is.

(this will allow us to deal with
black holes where the concept of an
observer at the origin gets tricky)

Note: • $v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$ in Newtonian gravity
 $v_{\text{esc}} = c$??

- at large distances from the mass ($r \rightarrow \infty$)
this reverts to metric for flat space
since $\frac{2GM}{rc^2} \rightarrow 0$
- value of c^2 keeps ~~this~~ effects small
until we get close to ~~is~~ M

Q

How can we tell that this describes a spherically symmetric situation?

A

Only the dt and dr terms are different from the flat space metric \Rightarrow spherical symmetry

Examples of using the metric

Spatial distance dl , measured at the same time, between two points on the same radial line near mass m ?

$$dl = \sqrt{-ds^2}$$

Measured at same time $\Rightarrow dt = 0$

Radial line $\Rightarrow d\phi = d\theta = 0$

We call this proper distance
because it is measured at the
same time

(cf proper time between 2 events
occurring at the same location)

$$ds^2 = - \frac{dr^2}{1 - \frac{2GM}{rc^2}}$$

$$dl = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

So the proper distance is greater
than the coordinate distance dr .

This is due to curvature of space

In the same way, actual hiking distance is always greater than the distance in map coordinates on a topo. map due to curvature of the terrain.

Can also work out time dilation (and so gravitational redshift) using this metric.

A clock at rest at radial coord. r will measure a proper time interval

$$d\tau = \frac{ds}{c} = dt \sqrt{1 - \frac{2GM}{rc^2}}$$

$d\tau$ is shorter, the closer you get to $r=0$ (time dilation)

This metric will allow us to calculate
the shortest distance between 2 points
in curved space outside M
("vacuum solution")

Black holes

Neutron degeneracy pressure will
only work up to $\sim 3M_{\odot}$ after
that, collapse is inevitable

Newtonian formula for escape
velocity

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

$$\text{set } v_{\text{esc}} = c \Rightarrow R = \frac{2GM}{c^2}$$

When

$$R = \frac{2GM}{c^2}$$

Schwarzschild radius
"event horizon"

Roots in $\sqrt{1 - \frac{2GM}{rc^2}}$ goes to zero

in Schwarzschild metric

Proper time $\rightarrow 0 \Rightarrow$ Schw. radius

see Kip Thorne : Black holes &

Falling into a black hole Time Warps

from two perspectives :

(i) watch from a safe distance :

Astronaut accelerates as falls, shining light once/sec.

light signals ~~arrive~~ arrive further apart (i) doppler redshift

(ii) special relativity time dilation

(iii) grav. time dilation

light becomes dimmer as energy of photons redshifts

At $\sim 2\frac{r_s}{c}$ Schw. radius from event horizon, the ~~time bet signals~~ \uparrow without limit

Signals become much less strong

Astronaut appears frozen in time

(ii) actually fall into BH

Enormous tidal forces as one closer to BH

After cross event horizon, no escape

Not possible for a particle to be at rest inside Schw. radius

$$\text{If } \omega_{\text{rest}} dr = d\phi = d\theta = 0$$

Schwarzschild metric becomes

$$ds^2 = c^2 dt^2 \left(1 - \frac{R_s}{R}\right) < 0 \text{ since}$$

$$R < R_s$$

But this interval is spacelike,
would require velocity $>c$.

Even photons are pulled toward
the singularity at center, so she
does not see it coming

Q

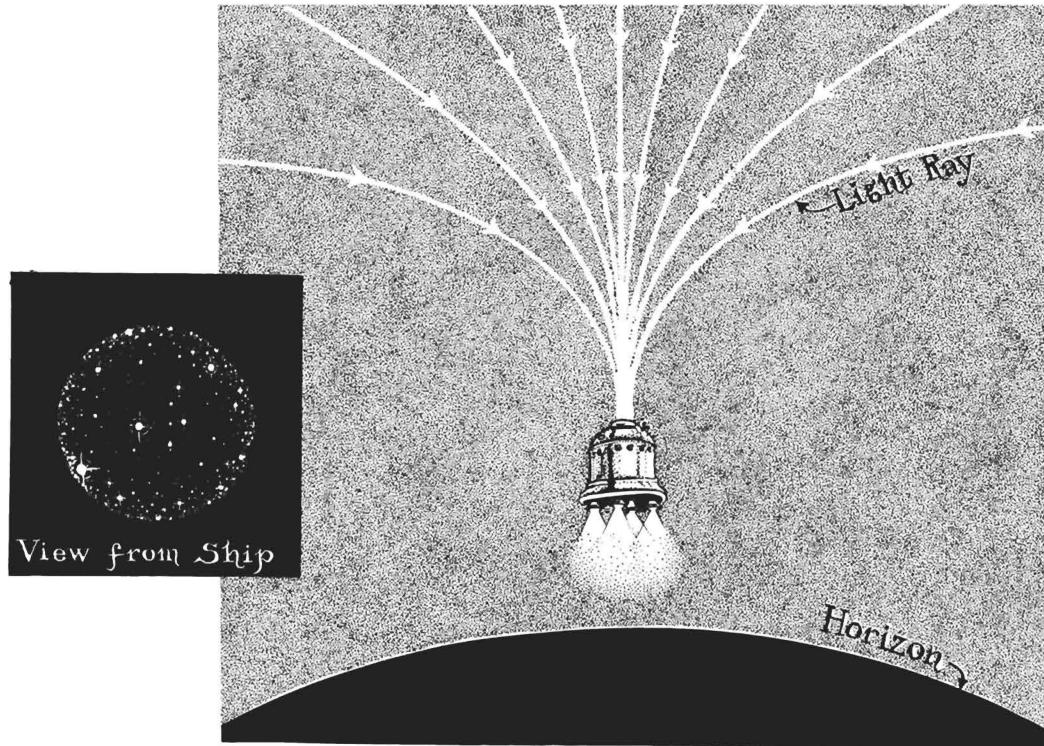
How might the universe appear to the astronaut if she turned her back on the black hole ?

A

light paths are bent by the black hole's gravity too

There is a point (the photon sphere) where it is possible for ~~the~~ photons to orbit the BH

P.4 The starship hovering above the black-hole horizon, and the trajectories along which light travels to it from distant galaxies (the light rays). The hole's gravity deflects the light rays downward ("gravitational lens effect"), causing humans on the starship to see all the light concentrated in a bright, circular spot overhead.



Q.

How might we detect an object
which is so massive that no
light can escape from it ?

A As a companion to a binary
star

X-rays given off by matter falling
into the black hole.

Detecting black holes

- measure mass of objects in binary system where only one star can be seen
If mass is large enough, it is a black hole
- detect accretion disk (X rays are good)
- gravitational lensing